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MATHEMATICAL EXPOSITIONS, No. 3

# THE THEORY OF POTENTIAL AND SPHERICAL HARMONICS

by

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and

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# PREFACE

THIS is a new book, not a translation from Sternberg's German book, *Potentialtheorie* in the "Goeschen" collection. A few chapters of the latter book have been used but these have been partly modified.

The book is designed chiefly for the use of students and teachers. The research worker will perhaps find some helpful suggestions, as well.

The student reading this book is presumed to have a thorough knowledge of differential and integral calculus. In some sections also an acquaintance with the most elementary theorems of the theory of analytic functions and of the theory of linear differential equations is desirable.

The text offers a short introduction to vector analysis and a presentation of the Fredholm theory of integral equations. The theory of spherical harmonics is also briefly explained. Consistent use of vector analysis is a characteristic feature of the book.

Among several books on potential theory we mention in particular O. D. Kellogg's *Foundations of Potential Theory* (1929) and G. C. Evans' *The Logarithmic Potential, Discontinuous Dirichlet and Neumann Problems* (1927), American Mathematical Society Colloquium Publications, vol. VI.

The authors hope that the present text fills a gap, by leading the student reader from the elements of potential theory to the solutions of boundary value problems in a simple and easily understandable way.

On account of space limitations it was not possible to deal with certain important topics. The methods of H. Poincaré, D. Hilbert, R. Courant, and other authors, for solving the boundary value problems under very general

conditions concerning the boundary of the regions, could not be treated, nor could the problems involving discontinuous boundary functions be discussed. In regard to this we refer the reader in particular to the paper of N. Wiener in the *Transactions of the American Mathematical Society*, vol. XXV (1923). Regarding the small section "Direct Methods of the Calculus of Variations" the reader would do well to extend his knowledge of this matter by studying R. Courant's chapter xx, "Variationsrechnung und Randwertprobleme" in Riemann-Weber *Die Differentialgleichungen der Mechanik and Physik*, vol. I (1925).

The authors wish to thank Dr. Carson Mark for his valuable assistance in the correction of proofs. They would appreciate any suggestions for improving the text.

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Cornell University,
Carnegie Institute of Technology,
October, 1943.

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# INTRODUCTION

## VECTORS AND SCALARS

Since the use of vector analysis is very convenient in the study of potential theory, we will first study the ideas of scalars and vectors as they appear in physics and the notation and simpler rules of vector operations and vector calculus.

A *scalar* is a quantity which is measured by a single number. Some examples are density, temperature, electrical charge density, viscosity of a fluid, etc. A scalar may have a constant value, or its value may vary from point to point in space and perhaps with the time.

A *vector* is a quantity which has magnitude and direction, as force, velocity, and acceleration. A vector can be geometrically represented by an arrow or directed line-segment in the direction of the vector, having a length (measured in centimetres, for example) equal to the number which represents (in any units) the magnitude of the vector.[1]

The orthogonal projection of a vector on any direction is called the *component* of the vector in this direction; this may be positive or negative. A vector $\boldsymbol{a}$ is evidently determined by its components $a_1$, $a_2$, $a_3$ in the direction of the $x$, $y$, $z$ axes of a rectangular coordinate system. The vector itself is geometrically represented by the interior diagonal of a rectangular box whose edges are the components.

The *magnitude* of a vector is

$$(1) \qquad a = |\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}\ ;$$

thus a letter in ordinary type represents the magnitude of the vector which is represented by the same letter in bold-face type.

[1]On account of their physical meaning, scalars and vectors are independent of the choice of coordinate axes.

The direction of the vector $\boldsymbol{a}$ is indicated by its *direction cosines*:

(2) $\cos(a, x) = a_1/a, \ \cos(a, y) = a_2/a, \ \cos(a, z) = a_3/a,$

the cosines of the angles which it makes with the axis directions.

The component of $\boldsymbol{a}$ in any direction $\boldsymbol{n}$ is

(3) $$a_n = a \cos(a, n),$$

and hence has its maximum value when $\boldsymbol{n}$ has the same direction as $\boldsymbol{a}$.

From analytic geometry, the formula for the cosine of the angle between the directions of $\boldsymbol{a}$ and $\boldsymbol{n}$ is

(4) $$\cos(a, n) = \cos(a, x) \cos(n, x) + \cos(a, y) \cos(n, y) + \cos(a, z) \cos(n, z).$$

If the two directions are perpendicular, this expression vanishes. The application of (4) to (3) gives

(5) $$a_n = a \, [\cos(a, x) \cos(n, x) + \ldots],$$

and from the formulas in (2) we get

(6) $$a_n = a_1 \cos(n, x) + a_2 \cos(n, y) + a_3 \cos(n, z).$$

According to (6) the component $a_n$ in an arbitrary direction $\boldsymbol{n}$ is represented linearly and homogeneously by the three direction cosines $\cos(n, x)$, $\cos(n, y)$, $\cos(n, z)$ of this direction, and the components $a_1$, $a_2$, $a_3$ in the directions of the axes are coefficients independent of $\boldsymbol{n}$. This representation of the component $a_n$ is obviously invariant under a transformation of the axes. That means: If $\xi$, $\eta$, $\zeta$ are a new system of axes, the equation

$$a_n = a_\xi \cos(n, \xi) + a_\eta \cos(n, \eta) + a_\zeta \cos(n, \zeta),$$

analogous to (6), holds true.

On the basis of (6) the concept of vector can be defined in another, more abstract but nevertheless very important, way. By (6) a number $a_n$ is associated with every direction $\boldsymbol{n}$ of the space. The vector $\boldsymbol{a}$ is then defined to be the association of

the numbers $a_n$ with the directions $\boldsymbol{n}$. The direction of the vector is defined to be that direction in which $a_n$ gets its maximum, so that the direction cosines of this direction are proportional to $a_1$, $a_2$, $a_3$, and the magnitude of the vector is defined to be the value of this maximum. It is obvious that direction and magnitude are again determined by (2) and (1) respectively. The reader should prove it!

The vector $-\boldsymbol{a}$ is defined to be the vector having the same magnitude as $\boldsymbol{a}$ but the direction opposite to that of $\boldsymbol{a}$; its components are therefore $-a_1$, $-a_2$, $-a_3$. The null-vector $\mathbf{0}$ is a vector whose three components are zero; its magnitude is therefore zero and its direction indeterminate.

Two vectors are said to be *equal* if and only if their corresponding components are equal respectively: $\boldsymbol{a} = \boldsymbol{b}$ when $a_i = b_i$ $(i = 1, 2, 3)$.

The *sum* or *resultant* of two vectors,

$$\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}, \tag{7}$$

is defined as the vector represented by the diagonal of the parallelogram having $\boldsymbol{a}$ and $\boldsymbol{b}$ as sides. The sum is independent of choice of axes, and is commutative. The sum has the components $a_1 + b_1$, $a_2 + b_2$, $a_3 + b_3$, so that the components are added algebraically. From (7) it follows that $\boldsymbol{b} = \boldsymbol{c} - \boldsymbol{a}$, by using the definition of $-\boldsymbol{a}$ and the interpretation that $\boldsymbol{c} - \boldsymbol{a}$ means $\boldsymbol{c} + (-\boldsymbol{a})$. There are no difficulties

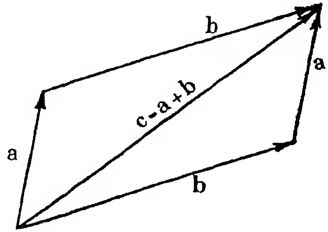


FIG. 1

in forming the sum of three vectors. It is easily seen that the associative law holds

$$\boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c},$$

so that the parentheses may be omitted and the sum written simply $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$. Of course the same statement holds for more than three vectors.

If $m$ is a scalar, then $m\boldsymbol{a}$ is a vector of magnitude $|m|a$ and in the same direction as $\boldsymbol{a}$ or in the opposite direction, according as $m$ is positive or negative; its components are $ma_1$, $ma_2$, $ma_3$. It is easily seen that

$$m(\boldsymbol{a} + \boldsymbol{b}) = m\boldsymbol{a} + m\boldsymbol{b}.$$

The *dot product* (*scalar product* or *inner product*) is defined by

$$\boldsymbol{a}\cdot\boldsymbol{b} = ab\cos(a, b) \tag{8}$$

and is evidently independent of the choice of coordinate axes. By the use of (2) and (4), it is seen that in terms of the rectangular components,

$$\boldsymbol{a}\cdot\boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3. \tag{9}$$

The dot product is evidently commutative, $\boldsymbol{a}\cdot\boldsymbol{b} = \boldsymbol{b}\cdot\boldsymbol{a}$ from its definition. It can be easily shown to be distributive when combined with addition, so

$$\boldsymbol{a}\cdot(\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\cdot\boldsymbol{c}. \tag{10}$$

The dot product is equal to the product of the magnitude of $\boldsymbol{a}$ by the component of $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$. The dot product of a vector by itself, written simply $\boldsymbol{a}^2$, is equal to $a_1^2 + a_2^2 + a_3^2$. The vanishing of the dot product is the condition for orthogonality of two vectors.

In vector analysis, it is frequently convenient to make use of the three unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ in the direction of the three

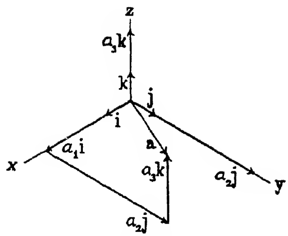


FIG. 2

axes. Then the vector $\boldsymbol{a}$ can be written as the sum of three component vectors in the axis directions,

It is frequently convenient also to use the following notation for a vector in terms of its components:

The *cross product* (*vector product*) $\boldsymbol{c}$ of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$,

(11) $$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b},$$

is a vector of magnitude equal to the area $ab \sin(a, b)$ of the parallelogram formed on $\boldsymbol{a}$ and $\boldsymbol{b}$, and having a direction perpendicular to their plane such that $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ form a right-hand axis system. That is, a rotation in their plane which carries the direction of $\boldsymbol{a}$ into that of $\boldsymbol{b}$ (rotation through an angle less than 180°) will appear to an observer facing in the direction of $\boldsymbol{c}$ to be a clock-wise or right-hand rotation (Fig. 3).

For the cross product, the commutative law does not hold, since from the definition it is evident that

$$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}.$$

It is not difficult to prove that the distributive law is valid:

$$\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}.$$

We will assume that the rectangular coordinate system used

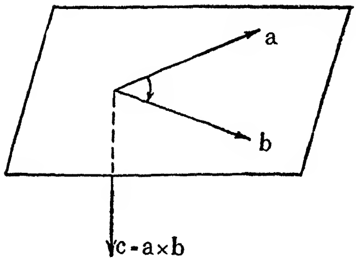


FIG. 3

is right-handed; then it is easily seen that, in terms of components,

$$(12) \qquad \begin{aligned} \boldsymbol{a} \times \boldsymbol{b} &= (a_2b_3 - a_3b_2)\,\boldsymbol{i} + (a_3b_1 - a_1b_3)\,\boldsymbol{j} + (a_1b_2 - a_2b_1)\,\boldsymbol{k} \\ &= (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1) \\ &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \end{aligned}$$

To prove this we note that

$$\begin{array}{lll} \boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k} & \boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i} & \boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j} \\ \boldsymbol{i} \times \boldsymbol{i} = 0 & \boldsymbol{j} \times \boldsymbol{j} = 0 & \boldsymbol{k} \times \boldsymbol{k} = 0. \end{array}$$

Therefore

$$\begin{aligned} \boldsymbol{a} \times \boldsymbol{b} &= (a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}) \times (b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}) \\ &= (a_1b_2)\boldsymbol{k} - (a_1b_3)\boldsymbol{j} - (a_2b_1)\boldsymbol{k} + (a_2b_3)\boldsymbol{i} + (a_3b_1)\boldsymbol{j} - (a_3b_2)\boldsymbol{i} \end{aligned}$$

in accordance with (12).

The gradient, divergence, and curl of a vector will be defined later as needed.

# CHAPTER I

## THE NEWTONIAN LAW OF GRAVITY

### Art. 1. The Newtonian Law

The Newtonian law of gravity states: *two concentrated masses $m_1$ and $m_2$ located at the points $P_1$ and $P_2$ exert on each other a force of attraction proportional to the product of their masses and inversely proportional to the square of the distance between them. The direction of the force is along the line joining the masses.* If we let $r$ and $F$ be the distance $P_1P_2$ and the magnitude of the force respectively, then

$$F = k\frac{m_1 m_2}{r^2},$$

where $k$ is called the gravitational constant. The force $m_1$ exerts on $m_2$ has the direction $P_2P_1$, while the force acting on $m_1$ has the direction $P_1P_2$.

The magnitude of the gravitational constant depends on the units of mass, distance and force selected. In the c.g.s. system with the force measured in dynes,

$$k = 6.66 \times 10^{-8}.$$

Hence two point-masses of one gram each at a centimetre distance apart attract each other with a force of 0.000,000,066,6 dynes.

It is customary to simplify the equations of potential theory by using a unit of force such that $k = 1$, so that

$$F = \frac{m_1 m_2}{r^2}.$$

This unit of force is called the gravitational unit of force. It is evidently equal to $6.66 \times 10^{-8}$ dynes, and hence is very small. Measuring force in this unit we have $F = 1$ when $m_1 = m_2 = 1$ and $r = 1$. We will mostly use this unit in this book (but not in the following example).

*Example*: to calculate the mean density of the earth. The length of the equator is 40,000 km. or $4 \times 10^9$ cm.; the acceleration of gravity is 981 cm/sec$^2$ on the earth's surface. In the law of force, let $m_1 = M$ be the mass of the earth concentrated in its centre (see Article 3) and let $m_2$ be 1 gm. at the earth's surface. Then $F = \frac{kM}{r^2}$ where $k = 6.66 \times 10^{-8}$ and $r = \frac{4 \times 10^9}{2\pi}$; moreover, $F = 981$ dynes, so that $981 =$

$r^2$ or $M$ $\frac{31r^2}{k}$. If we let $\mu$ be the mean density of the earth, then $M = \frac{4\pi r^3 \mu}{3}$. Equating the two expressions for $M$,

$$\mu = \frac{981 r^2}{k} \cdot \frac{3}{4\pi r^3} = \frac{3 \times 981}{4\pi k r}$$

$$= 5.5 \text{ approx.}$$

In electrostatics, Coulomb's Law has the same form as Newton's gravitational law. Two point-charges of electricity $e_1$ and $e_2$ at the points $P_1$ and $P_2$ at a distance $r$ apart exert on each other a force

$$\frac{e_1 e_2}{r^2}.$$

The force is along the line $P_1P_2$; like charges repel, and charges of unlike sign attract. Here it is customary to use the dyne as the unit of force, and to select the unit of electrical charge so that the constant of proportionality is unity in Coulomb's Law. That quantity of electricity which exerts a force of one dyne on an equal quantity of electricity at a distance of one centimetre is called an *electrostatic unit* (c.g.s.e.s.u.) *of electricity*.

## Art. 2. The Force Field

The force which would act on a unit mass held at any point in space is called the value of the force field at that point. We will first study the force field due to a concentrated mass.

Let the mass $m$ be concentrated at the point $Q:(x_1, y_1, z_1)$, and consider the force which this exerts on a unit mass at $P:(x, y, z)$. Let the vector from $P$ to $Q$ be denoted by $\boldsymbol{r}$,

$$\boldsymbol{r} = (x_1 - x)\,\boldsymbol{i} + (y_1 - y)\,\boldsymbol{j} + (z_1 - z)\,\boldsymbol{k},$$

then the force acting on the unit mass at $P$ is in the direction of $\boldsymbol{r}$ or in the direction of the unit vector $\dfrac{\boldsymbol{r}}{r}$; so that the force vector is equal to $\dfrac{m}{r^2}\dfrac{\boldsymbol{r}}{r}$, or

$$\boldsymbol{F} = \frac{m\boldsymbol{r}}{r^3}.$$

This vector is the value of the force field at $P$ due to the mass $m$ concentrated at $Q$. This field has the components

$$X = \frac{m(x_1 - x)}{r^3},\quad Y = \frac{m(y_1 - y)}{r^3},\quad Z = \frac{m(z_1 - z)}{r^3}.$$

When several masses $m_1, m_2, \ldots . m_n$ are located at the points $Q_1, Q_2, \ldots, Q_n$ respectively, where $Q_s$ is $(x_s, y_s, z_s)$, they produce a force field or force on a unit mass at $P:(x, y, z)$ which is the resultant of the forces exerted separately by the individual masses. The force due to the mass $m_s$ is

$$\boldsymbol{F}_s = \frac{m_s \boldsymbol{r}_s}{r_s^3},$$

where $\boldsymbol{r}_s = (x_s - x, y_s - y, z_s - z)$. Adding these together, the total field at $P$ is

$$\boldsymbol{F} = \sum_{s=1}^{n} \frac{m_s \boldsymbol{r}_s}{r_s^3},$$

with components

$$X = \sum_{s=1}^{n} \frac{m_s(x_s - x)}{r_s^3}, \quad Y = \sum_{s=1}^{n} \frac{m_s(y_s - y)}{r_s^3}, \quad Z = \sum_{s=1}^{n} \frac{m_s(z_s - z)}{r_s^3}.$$

We consider next a continuous distribution of mass, which occupies a bounded region $V$. Let the element of volume $dV = d\xi d\eta d\zeta$ at the point $Q{:}(\xi, \eta, \zeta)$ of $V$ contain the mass $dm$. Then the field of force at $P{:}(x, y, z)$ due to $dm$ is

$$\frac{\boldsymbol{r}\,dm}{r^3}, \qquad \boldsymbol{r} = (\xi - x, \eta - y, \zeta - z).$$

It is at first assumed that $P$ is outside the volume $V$. The total force at $P$ is

$$\boldsymbol{F} = \iiint\limits_V \frac{\boldsymbol{r}\,dm}{r^3} =$$
$$\left(\iiint\limits_V \frac{(\xi - x)dm}{r^3}, \iiint\limits_V \frac{(\eta - y)dm}{r^3}, \iiint\limits_V \frac{(\zeta - z)dm}{r^3}\right).$$

If $\rho = \rho(\xi, \eta, \zeta)$ is the density, then $dm = \rho dV$ and

$$\boldsymbol{F} = \iiint\limits_V \frac{\boldsymbol{r}\rho dV}{r^3}.$$

The density is a scalar function which we shall assume to be bounded and integrable.

If the mass is distributed over a surface (plane or curved) with the surface density $\sigma$, the force is

$$\boldsymbol{F} = \iint\limits_S \frac{\boldsymbol{r}\sigma dS}{r^3},$$

where $dS$ is the element of area of the surface $S$. Finally if the mass is on a curve $C$ with a linear density $\gamma$, the force field it produces is

$$F = \int_C \frac{r\gamma ds}{r^3},$$

where $ds$ is the element of arc of the curve $C$.

The field $F$ due to a space distribution of mass has a meaning when $P$ is inside the region $V$ as well as when $P$ is an exterior point. The integral

$$F = \iiint_V \frac{r\rho dV}{r^3}$$

is an improper integral when $P$ is inside $V$, because $r \to 0$ as $Q \to P$; but if we transform this integral by introducing spherical coordinates for $Q$ with $P$ as origin, then

$$dV = r^2 \sin\theta\, dr\, d\theta\, d\phi, \text{ and } (\xi - x) = r \sin\theta \cos\phi,$$

so that
$$X = \iiint_V \rho \sin^2\theta \cdot \cos\phi\, dr\, d\theta\, d\phi$$

which is not an improper integral. Similarly it can be seen that the integrals for $Y$ and $Z$ become proper integrals in the new coordinates. Hence $F$ exists and is defined by these integrals at interior points of $V$.

## Art. 3. Examples and Exercises

1. A spherical shell or surface of radius $a$, having a surface mass distribution of density $\sigma$. The distribution is supposed to be homogeneous, so that $\sigma$ = constant.

Let the point $P$ be taken on the positive $z$-axis. On account of the symmetry, $Y$ and $X$ are zero. Show that

$$Z = \begin{cases} -\dfrac{4\pi a^2\sigma}{z^2} = -\dfrac{M}{z^2} & \text{for } z > a, \\ 0 & \text{for } z < a. \end{cases}$$

The force field at exterior points is therefore the same as if the mass were concentrated at the centre. (From example 6, below, $Z = -2\pi\sigma$ when $z = a$.)

2. A homogeneous spherical solid of uniform density $\rho$ and radius $a$.

Take $P$ on the $z$-axis above the sphere, and show that

$$Z = -\frac{4\pi a^3\rho}{3z^2} = -\frac{M}{z^2}.$$

The solid sphere therefore attracts at exterior points as if its mass were concentrated at the centre.

Show that this is also true if the density $\rho$ is a function of the distance from the centre of the sphere.

2a. A hollow thick spherical shell or solid filling the space between two concentric spheres, with density constant or a function of the distance from the centre. Show that the force is zero for a point inside the inner sphere, and is the same as if the mass were concentrated at the centre for a point outside the outer sphere.

3. Homogeneous straight wire $AB$ of length $l$ and density $\gamma$.

Take the wire as the $z$-axis with $A$ at the origin and $B$ at $z = l$. Take $P$ on the $z$-axis; then evidently $X = Y = 0$.

For $z > l$, $\quad Z = \int_0^l \frac{(\zeta - z)\gamma d\zeta}{r^3}$, where $r = |\zeta - z| = z - \zeta$;

$$Z = -\gamma \int_0^l \frac{(\zeta - z)d\zeta}{(\zeta - z)^3} = -\frac{\gamma l}{z(z - l)}.$$

For $z < 0$, $\quad Z = \int_0^l \frac{(\zeta - z)\gamma d\zeta}{r^3} = +\frac{\gamma l}{z(z - l)}$

On approaching the end-points $z = 0$ or $z = l$, $Z$ becomes

infinite. When $P$ is an interior point $(0 < z < l)$, we can take $P$ as the mid-point for a small segment of wire; by symmetry this exerts no force on $P$. But $P$ is an exterior point for the remainder of the wire, so that the total force can be found.

4. Infinite homogeneous wire.

Let the wire lie along the $z$-axis, and take $P$ on the positive $x$-axis. Then by symmetry $Y = Z = 0$, and

$$
\begin{aligned}
X &= \int_{-\infty}^{\infty} \frac{\gamma \cos \alpha d\zeta}{r^2}, \text{ where } \cos \alpha = \frac{-x}{r}, r^2 = x^2 + \zeta^2, \\
&= -\gamma x \int_{-\infty}^{\infty} \frac{d\zeta}{(x^2 + \zeta^2)^{3/2}} \\
&= -\frac{\gamma}{x} \int_{-\infty}^{\infty} \frac{du}{(1 + u^2)^{3/2}}, \quad u = \frac{\zeta}{x}, \\
&= -\frac{2\gamma}{r}.
\end{aligned}
$$

The force is therefore inversely proportional to the distance from the wire. This can be used as an interpretation of the logarithmic potential (Chapter 2, Article 3).

5. Homogeneous disk of radius $a$.

Let the disk lie in the $xy$-plane with centre at the origin, and take $P$ on the $z$-axis. By symmetry, $X = Y = 0$, and

$$
\begin{aligned}
Z &= \iint_S \frac{(\zeta - z)\, \sigma dS}{r^3} \quad (\zeta = 0; r^2 = \xi^2 + \eta^2 + z^2) \\
&= -\sigma z \iint \frac{dS}{(\xi^2 + \eta^2 + z^2)^{3/2}}.
\end{aligned}
$$

By introducing polar coordinates $\xi = \rho \cos \theta$, $\eta = \rho \sin \theta$,

$$Z = -\sigma z \int_0^{2\pi} \int_0^a \frac{\rho d\rho d\theta}{(z^2+\rho^2)^{3/2}}$$

$$= -2\pi\sigma z \left\{ \frac{1}{|z|} - \frac{1}{\sqrt{a^2+z^2}} \right\}$$

$$= \begin{cases} -2\pi\sigma + \dfrac{2\pi\sigma z}{\sqrt{a^2+z^2}} \text{ for } z > 0, \\ 2\pi\sigma + \dfrac{2\pi\sigma z}{\sqrt{a^2+z^2}} \text{ for } z < 0. \end{cases}$$

We find from this that

$$Z_+ = \lim_{\to +0} Z = -2\pi\sigma; \ Z_- = \lim_{z \to -0} Z = 2\pi\sigma.$$

Also $Z(0) = 0$, so that

$$Z(0) = \frac{Z_+ + Z_-}{2}.$$

6. Homogeneous spherical surface; to calculate $\boldsymbol{F}$ for $P$ on the surface.

Let the radius be $a$, centre at the origin, and let $\sigma$ be the surface density. Take $P$ at $(0, 0, a)$; then the tangential components $X$, $Y$ vanish by symmetry, and the normal component

$$Z = \sigma \iint_S \frac{\zeta - a}{r^3} dS$$

This integral is improper, but is convergent and can easily be evaluated. By using spherical coordinates, it becomes

$$Z = \sigma \int_0^{\pi} \int_0^{2\pi} \frac{(a \cos\theta - a)\, a^2 \sin\theta d\phi d\theta}{[a^2 \sin^2\theta + a^2(1-\cos\theta)^2]^{3/2}}$$

$$= -2\pi\sigma \int_0^{\pi} \frac{\sin\theta d\theta}{2^{3/2}(1-\cos\theta)^{1/2}}$$

$$= -2\pi\sigma.$$

Hence the force is independent of $a$. From example 1, we

find that for the forces just outside and just inside the surface

$$Z_+ = -4\pi\sigma, \quad Z_- = 0,$$

so that

$$Z(P) = \frac{Z_+ + Z_-}{2}.$$

7. To find $\boldsymbol{F}$ in the interior of a homogeneous solid sphere.

Let $\rho$ be the density, $a$ the radius, and $S$ the surface; let $OP = \boldsymbol{R}$ be the vector from the centre to the point $P$ inside the sphere.

Let $S_1$ and $S_2$ be concentric spherical surfaces with radii $a_1 > R$ and $a_2 < R$. That portion of the sphere between $S_1$ $S$ exerts no force at $P$ (Example 2a). The mass inside the surface $S_2$ exerts a force (Example 2)

$$\boldsymbol{F} = -\frac{4\pi a_2{}^3\rho}{3R^2}\frac{\boldsymbol{R}}{R}.$$

By letting $a_1 \to R$ and $a_2 \to R$, it is seen that

$$\boldsymbol{F} = -\frac{4\pi\rho}{3}\boldsymbol{R}.$$

The force is therefore proportional to the distance $R$ from the centre. This method can also be used when $\rho$ is not a constant but varies with the distance from the centre of the sphere.

## Art. 4. Force Fields. Lines of Force. Vector Fields. Velocity Fields

If a force is defined at every point in space or in a portion of space, the space possesses a physical property and is called a *field of force*, in accordance with Art. 2. For example, a space distribution of mass would exert on a unit point-mass at any point $P$ a force, which is the value at $P$ of the force field caused by the distribution of mass. Similarly, masses distributed over surfaces or curves produce force fields. The force

field produced by a point-mass is not defined at the point itself, because the force becomes infinite as this point is approached.

A *line of force* is a curve which has, at each of its points, the same direction as the force at that point. The lines of force due to a point-mass are the straight lines through that point where the mass is located; and the lines of force due to a homogeneous spherical mass are the straight lines through the centre of the sphere. In general the lines of force are, according to their definition, the solutions of a system of ordinary differential equations,

$$\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z},$$

where (X, Y, Z) is the force field and $(dx, dy, dz)$ is a displacement along the line of force. Two arbitrary constants enter in the integration; hence the lines of force form a two-parameter family of curves.

*Examples*

1. For a mass-point $m$ located at $Q$ (taken as the origin),

$$\boldsymbol{F} = -\frac{m\boldsymbol{r}}{r^3}, \text{ where } \boldsymbol{r} = (x, y, z).$$

Hence
$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

are the differential equations of the lines of force, with the solutions $y = ax$, $z = bx$. These equations represent the lines of force, which are straight lines through the origin.

2. Let the field be $\boldsymbol{F} = (x, y, -z)$. Then

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{-z},$$

which have the solution $y = ax$, $zx = b$.

A point where $\boldsymbol{F} = \boldsymbol{0}$ or $X = Y = Z = 0$ is a point of equilibrium. Such a point is in general a singular point for the differential equations of the lines of force.

If an arbitrary vector is defined at each point in space or in a portion of space, this space is called a *vector field*. The lines corresponding to the lines of force are the *field-lines*.

For example, consider the motion of a fluid. Let $(x, y, z)$ be the coordinates of a particle at the time $t$, then the motion is associated with the differential equations

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{v}(x, y, z, t)$$

where $\boldsymbol{v}$ is the velocity of a particle. The field $\boldsymbol{v}$ is a *velocity field*. The velocity in general depends on the time $t$ as well as on the place $(x, y, z)$. When the velocity is independent of $t$ explicitly, the flow is called stationary.

The solution of the above differential equation gives

$$\boldsymbol{r} = \phi(t, a, b, c) \text{ or}$$

$$x = \phi_1(t, a, b, c), \quad y = \phi_2(t, a, b, c), \quad z = \phi_3(t, a, b, c).$$

A particular choice of numerical values for $a$, $b$, $c$ corresponds to a particular particle; the above equation $\boldsymbol{r} = \phi(t, a, b, c)$ therefore represents the paths of particular particles. The parameters may be so chosen that they are the coordinates of the particle in question at the beginning of the motion, at $t = 0$. We designate them in this case by $x_0$, $y_0$, $z_0$; then

$$\boldsymbol{r} = \phi(t, \boldsymbol{r}_0).$$

Equations of motion in this form are known as Lagrange equations.

A field-line, which is here called a flow-line or stream-line, is by definition such that at each point of the line (at some particular instant) the line has the direction of the velocity at that point. The differential equations of the stream-lines are therefore

$$\frac{dx}{v_1(x, y, z, t)} = \frac{dy}{v_2(x, y, z, t)} = \frac{dz}{v_3(x, y, z, t)},$$

where $t$ is considered constant. On the other hand, from the vector equation above it follows that the paths of particles are the solutions of

$$\frac{dx}{v_1} = \frac{dy}{v_2} = \frac{dz}{v_3} = dt$$

where $t$ is the independent variable. Thus it is apparent that the paths of particles are in general different from the stream-lines. They agree if and only if the ratios $v_1: v_2: v_3$ are independent of the time. In particular, when $\boldsymbol{v}$ is independent of $t$ or the flow is stationary, the stream-lines are the same as the actual paths of the particles.

*Examples*

1. $\boldsymbol{v} = (xt, y, z)$.

This velocity field has the paths of particles

$$(x, y, z) = (c_1 e^{t^2/2}, c_2 e^t, c_3 e^t),$$

and the stream-lines

$$x = a y^t, \quad y = bz$$

at any fixed instant $t$ = constant.

2. $\boldsymbol{v} = (5, t, 0)$.

The trajectories of the particles are the parabolas

$$\begin{cases} x = 5t + c_1 \\ y = c_2 t^2, \\ z = c_3, \end{cases}$$

while the flow-lines at any instant are the straight lines

$$y = \frac{t}{5} x + a,$$

$$z = b.$$

# CHAPTER II

## CONCEPT OF POTENTIAL

### Art. 1. Work. Potential. Gradient of a Scalar

We will now show how the idea of potential arises in mechanics. The motion of a mass-particle in a force field $\boldsymbol{F} = (X, Y, Z)$ is a simple example. We will assume that the field is a continuous function of position in space, and for simplicity that the mass-particle has unit mass. By Newton's second law of motion, the motion is governed by

$$\boldsymbol{F} = \boldsymbol{r}'', \text{ or } X = x'', Y = y'', Z = z'',$$

where the primes mean differentiation with respect to time. Here $\boldsymbol{r} = (x, y, z)$ is the vector from the origin to the particle. Let $\boldsymbol{v}$ be the velocity of the particle. Then

$$\tfrac{1}{2} \frac{d}{dt} \boldsymbol{v}^2 = \boldsymbol{r}'' \cdot \boldsymbol{r}' = \boldsymbol{F} \cdot \boldsymbol{v};$$

but

$$(1) \qquad E(t) = \tfrac{1}{2} v^2 = \tfrac{1}{2} \boldsymbol{v}^2$$

is the kinetic energy of the particle. It is a scalar independent of the coordinate system. From the above equations

$$\frac{dE}{dt} = \boldsymbol{F} \cdot \frac{d\boldsymbol{r}}{dt},$$

and by integration with respect to the time,

$$(2) \qquad E(t) - E(t_0) = \int_{P_0}^{P} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{P_0}^{P} (X dx + Y dy + Z dz),$$

where the particle moves along a certain curve from $P_0$ to $P$ in the time from $t_0$ to $t$. (We consider here, and later, curves which possess a continuously turning tangent except perhaps for a finite number of points.) The expression on the left of

(2) is the increase in kinetic energy of the unit mass-particle from $t_0$ to $t$. By the definition of the dot-product, the integrand on the right side of (2) is $\boldsymbol{F}\cdot d\boldsymbol{r} = F \cos \theta \, ds = F_s ds$, the element of work which the force performs on the portion $ds$ of the path. The line integral $U = \int_{P_0}^{P} \boldsymbol{F}\cdot d\boldsymbol{r}$ is therefore the total work performed during the motion.

The work is in general dependent not merely on the position of the points $P_0$ and $P$, but also on the path of the particle between them. However, the work is *independent of the path L* used, if the integral has the same value for all paths, from $P_0$ to $P$, which can be deformed continuously into each other without leaving the force field.

We now impose the condition that the integral

(3) $$\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = 0$$

for any closed path $C$ which can be shrunk to a point without leaving the field. A field which satisfies this condition is called *conservative*. Then the integral $U$ for any two points of the field is independent of the path between them,[1] and conversely property (3) follows from the property of independence of the value of the integral on the path between end-points. When we consider $P_0$ as fixed and $P$ as variable, the integral $U = \int_{P_0}^{P} \boldsymbol{F}\cdot d\boldsymbol{r}$ represents a function of $P$ (naturally independent of the choice of coordinate system). This scalar function

(4) $$U(P) = \int_{P_0}^{P} \boldsymbol{F}\cdot d\boldsymbol{r}$$

is called the *potential* of the field $\boldsymbol{F}$. We will now always assume that the field is a simply-connected region, that is, that

[1]The integral has the same value at least for any pair of paths, whose combination is a closed curve which can be continuously shrunk to a point without leaving the field.

every closed curve lying in the field can be continuously shrunk to a point without leaving the field. (For example, a cube, sphere or cylinder, a sphere with one or more inner points removed, or the region between two concentric spherical surfaces is simply-connected, but a torus is not a simply-connected region.)

For a simply-connected region, the potential $U$ (with $P_0$ fixed) is a single-valued function of $P$, because the integral (4) has the same value for every curve of the field joining these points. If we take a different fixed point $P_1$ instead of $P_0$, then

$$U_1(P) = \int_{P_1}^{P} \boldsymbol{F}\cdot d\boldsymbol{r} = \int_{P_1}^{P_0} \boldsymbol{F}\cdot d\boldsymbol{r} + \int_{P_0}^{P} \boldsymbol{F}\cdot d\boldsymbol{r},$$

so that $U_1$ and $U$ only differ by a constant, namely $\int_{P_1}^{P_0} \boldsymbol{F}\cdot d\boldsymbol{r}$. The potential $U$ is therefore uniquely determined by the field $\boldsymbol{F} = (X, Y, Z)$ except for an arbitrary additive constant.

From (4), by the rules of calculus, we get

$$\frac{\partial U}{\partial x} = X, \quad \frac{\partial U}{\partial y} = Y, \quad \frac{\partial U}{\partial z} = Z,$$

or more briefly

$$\boldsymbol{F} = \text{grad } U. \tag{5}$$

(See the end of this article for a discussion of the gradient as used in vector analysis.) The derivatives of the potential in the axis directions are therefore the force components in these directions; hence, conversely, the field is determined by its potential. Similarly, the directional derivative of the potential in any direction is the force component in this direction, since

$$\frac{\partial U}{\partial n} = \frac{\partial U}{\partial x}\frac{dx}{dn} + \frac{\partial U}{\partial y}\frac{dy}{dn} + \frac{\partial U}{\partial z}\frac{\partial z}{\partial n} =$$

$$\frac{\partial U}{\partial x}\cos(n,x) + \ldots = X\cos(n,x) + \ldots = \boldsymbol{F}\cdot\boldsymbol{n} = F_n.$$

We have just seen that a conservative field has a potential such that $\boldsymbol{F} = \text{grad } U$. Conversely, if it is assumed that a field has a potential such that $\boldsymbol{F} = \text{grad } U$, then it is conservative; for it follows that for any two points $P_0$ and $P$ of the field

$$\int_{P_0}^{P} \boldsymbol{F}\cdot d\boldsymbol{r} = \int_{P_0}^{P} \left( \frac{\partial U}{\partial x} dx + \dots \right) = \int_{P_0}^{P} dU = U(P) - U(P_0),$$

so that the line integral is independent of the path and hence vanishes for closed curves.

The integral $-U = -\int_{P_0}^{P} \boldsymbol{F}\cdot d\boldsymbol{r}$ is the *potential energy*; it is the work which must be done to bring the particle from $P$ to $P_0$. Equation (2) is then the equation for the conservation of energy for the single particle. It shows that the sum of the potential and kinetic energies is a constant.

In electrical fields governed by Coulomb's law where elements of like sign repel, it is usual to define the potential $U$ by the relation

$$\text{grad } U = -\boldsymbol{F}$$

instead of (5). The potential is then equal to the potential energy, while in gravitational fields it is the negative of the potential energy.

If the field extends to infinity and vanishes to a sufficient order, then the point $P_0$ in (4) can be taken at infinity, thus fixing the undetermined constant. The potential at $P$ is then the work which is performed by the field to bring the mass from infinity to $P$; this is equivalent with the condition $U = 0$ at infinity.

From a purely mathematical point of view, we note that if $X$, $Y$, $Z$ are continuous functions of $x$, $y$, $z$, then $\boldsymbol{F} = \text{grad } U$ is a system of three simultaneous differential equations for the function $U$. Such a system has in general no solution. From

the above considerations it follows that (3) is an integrability condition for $\boldsymbol{F} = \text{grad } U$—that is, that these differential equations have a solution if and only if condition (3) is satisfied. The solution is given by (4) and is determined except for an additive constant.

Under the assumption that $X$, $Y$, $Z$ are continuously differentiable, it follows from (5) that

$$\text{(6)} \qquad \text{curl } \boldsymbol{F} = \boldsymbol{0}, \text{ i.e. } \frac{\partial Z}{\partial y} = \frac{\partial Y}{\partial z}, \frac{\partial X}{\partial z} = \frac{\partial Z}{\partial x}, \frac{\partial Y}{\partial x} = \frac{\partial X}{\partial y}.$$

(See end of article and Chapter 3, Art. 5.) These three equations on $\boldsymbol{F}$ are therefore necessary conditions for the existence of a potential; we will find (Chapter 3, Art. 5) that they are also sufficient.

*Gradient and Curl*

The derivation of a field vector $\boldsymbol{F}$ from its potential is an example of the use of the gradient of a scalar, which we will pause to study.

Let $W = W(x, y, z)$ be a scalar function, continuously differentiable, as, for example, the temperature at any point in a body. The derivative in the direction of any unit vector $\boldsymbol{n}$ is

$$\text{(7)} \qquad \frac{\partial W}{\partial n} = \frac{\partial W}{\partial x} \cos (n, x) + \frac{\partial W}{\partial y} \cos(n, y) + \frac{\partial W}{\partial z} \cos (n, z).$$

The vector $\boldsymbol{A}$ whose components $\frac{\partial W}{\partial x}, \frac{\partial W}{\partial y}, \frac{\partial W}{\partial z}$ enter (7) is known as the gradient of $W$, and denoted by

$$\boldsymbol{A} = \text{grad } W = \nabla W \text{ (read “del” } W),$$

where $\nabla$ is the symbolic vector operator $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$. Since $\cos(n, x)$, etc., are the components of the unit vector $\boldsymbol{n}$, (7) is in the form of a dot product, and may be written

$$\frac{\partial W}{\partial n} = \nabla W \cdot \boldsymbol{n} = (\text{grad } W)_n = |\text{grad } W| \cos (A, n).$$

Hence the directional derivative is a maximum when it is in the direction $\boldsymbol{A}$, and the maximum value of the directional derivative is the magnitude of $\boldsymbol{A}$. These considerations show the physical meaning of the gradient and make it apparent that it is independent of the choice of axes (compare Introduction, abstract definition of vector).

Other uses of $\nabla$ are in the quantities defined by

(8) $\quad \text{div } \boldsymbol{F} = \nabla \cdot \boldsymbol{F} = \dfrac{\partial X}{\partial x} + \dfrac{\partial Y}{\partial y} + \dfrac{\partial Z}{\partial z}$ (divergence of $\boldsymbol{F}$) and

$$\text{curl } \boldsymbol{F} = \text{rot } \boldsymbol{F} = \nabla \times \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ X & Y & Z \end{vmatrix}$$

## Art. 2. Newtonian Potential of a Body

*A Newtonian field is conservative.*

Consider first the field which a point-mass $m$ at $Q:(x_1, y_1, z_1)$ produces. We have seen that this produces the field

$$\boldsymbol{F} = \frac{m\boldsymbol{r}}{r^3}, \quad \text{where } \boldsymbol{r} = (x_1 - x, y_1 - y, z_1 - z).$$

Evidently this field is the gradient of the function

$$U = \frac{m}{r} + c,$$

which is therefore the potential function of the field, since

$$\text{grad } U = \left(\frac{\partial U}{\partial x}, \ldots\right) = \left(\frac{m(x_1 - x)}{r^3}, \ldots\right) = \boldsymbol{F}.$$

The field extends to infinity, and if we make the function

unique by the condition $U = 0$ at $r = \infty$, we have finally $c = 0$. Thus the function

$$U = \frac{m}{r} \tag{9}$$

is called the Newtonian potential for a point-mass $m$.

Similarly, if several masses $m_s$ are located at the points $(x_s, y_s, z_s)$, then their potential is

$$U = \sum_{s=1}^{n} \frac{m_s}{r_s}, \quad \boldsymbol{r}_s = (x_s - x, y_s - y, z_s - z), \tag{10}$$

since it is easily seen that the correct expression for the force field is

$$\boldsymbol{F} = \operatorname{grad} U = \left( \sum_{s=1}^{n} \frac{m(x_s - x)}{r_s^3}, \ldots \quad . \right). \tag{11}$$

The potential function is continuous, together with its derivatives of all orders, except when $P$ is coincident with one of the source-points where a mass is located.

The problems of one or several point-masses are mere abstractions; in practice we usually have continuous distributions of mass. Suppose that we have a continuous distribution of mass filling a region $V$ of space (for example, the space inside a closed sphere). Let $P{:}(x, y, z)$ lie outside of $V$, and let $dm$ be the mass of the element of volume $dV$ = located at $Q{:}(\xi, \eta, \zeta)$.

The potential for this continuous distribution is

$$U = \iiint \frac{dm}{r}, \text{ where } \boldsymbol{r} = (\xi - x, \eta - y, \zeta - z), \tag{12}$$

since, by the rules of calculus, differentiation of this function leads to the correct law of force for the field due to the distribution,

$$\boldsymbol{F} = \operatorname{grad} U = (X, Y, Z) = \left( \iiint \frac{\xi - x}{r^3}\, dm, \ldots, \ldots \right). \tag{13}$$

The potential $U$ may be differentiated not merely once, but as often as desired with respect to $x$, $y$, or $z$; for, since $P$ is assumed outside $V$, the integrand is continuous and differentiable as often as desired with respect to $x$, $y$, $z$, and since the boundary surface is fixed, differentiation under the integral sign is permissible. If we introduce the density $\rho$, the above equations become

$$U = \iiint_V \frac{\rho dV}{r}, \tag{12*}$$

and

$$F = \text{grad } U = \left( \iiint \frac{(\xi - x)\rho dV}{r^3}, \ldots, \ldots \right). \tag{13*}$$

The density does not need to be a continuous function; it is sufficient to assume that it is bounded and integrable in $V$.

The potential defined by (12) or (12*) is called the *Newtonian potential* to distinguish it from the logarithmic potential which we will now discuss. We note again for emphasis that *the Newtonian potential is a continuous and arbitrarily often differentiable function of $P$:$(x, y, z)$ for all points $P$ outside $V$.*

### Art. 3. Logarithmic Potential

Let two points in the plane, $P$:$(x, y)$ and $Q$:$(\xi, \eta)$, be given, which attract each other with a force $F$ acting along the line $PQ$ and given by the law

$$F = k\,\frac{mm'}{r},$$

where $r = \sqrt{(\xi - x)^2 + (\eta - y)^2}$

is the distance between the points. Here $m$ and $m'$ may be called again the "masses" of $P$ and $Q$ respectively, and $k$ is a constant independent of the position and masses of the points. The essential difference between the Newtonian law of force and the present one is that *the force is now assumed to be inver-*

*sely proportional to the first power of the distance instead of to its square.* The force which $Q$ exerts on $P$ is

$$\mathbf{F} = (F\cos\alpha,\ F\cos\beta), \tag{14}$$

where $\alpha$ and $\beta$ are the angles which the direction from $P$ to $Q$ makes with the coordinate axes. If we let $km' = 1$, we have

$$\mathbf{F} = \frac{m\mathbf{r}}{r^2} = \left(\frac{m(\xi - x)}{r^2},\ \frac{m(\eta - y)}{r^2}\right), \tag{14*}$$

$$\text{where } \mathbf{r} = (\xi - x,\ \eta - y).$$

All the formulas of vector analysis are valid in 2-space (or in $n$-space) except those which involve the cross-product. The force field here is again the gradient (2-dimensional) of a potential function:

$$\mathbf{F} = \operatorname{grad} U = \left(\frac{\partial U}{\partial x},\ \frac{\partial U}{\partial y}\right), \tag{15}$$

where

$$U = m \log\frac{1}{r}. \tag{16}$$

This function is called the *logarithmic potential* due to the point-mass $m$ at $Q$, on account of the form of the function.

The force field

$$\mathbf{F} = \sum_{s=1}^{n} \frac{m_s\mathbf{r}_s}{r_s^2}, \quad \mathbf{r}_s = (\xi_s - x,\ \eta_s - y), \tag{17}$$

due to the masses $m_s$ at the points $Q_s:(\xi_s, \eta_s)$, has the potential

$$U = \sum_{s=1}^{n} m_s \log\frac{1}{r_s}. \tag{18}$$

A distribution of density $\sigma$ spread over a finite portion $S$ of the plane, attracting by the inverse first power law, produces a field at an external point $P$ given by

$$\mathbf{F} = \iint_S \frac{\sigma\mathbf{r}}{r^2}\,dS. \tag{19}$$

This force field has the potential function

$$U = \iint_S \log \frac{1}{r}\, \sigma dS. \tag{20}$$

In all cases, we have $\boldsymbol{F} = \text{grad } U$.

The logarithmic potential plays the same part in the plane as the Newtonian potential in space. It is a continuous and arbitrarily often differentiable function of the position of the point $P$, when this is varied outside the region $S$ of the plane where there is mass present.

## Art. 4. Newtonian Potential of Surface Distributions and of Double Layers

Let a plane or curved surface $S$ carry a surface distribution of mass of density $\sigma$ which attracts by the Newtonian inverse square law. Assume the surface to be finite, with continuously turning tangent plane or made of a finite number of pieces with continuously turning tangent plane, joined along edges; also assume there are only a finite number of corners or sharp points (as the vertex of a cone).

The force field and potential are here given by double integrals extended over $S$

$$\boldsymbol{F} = \left( \iint_S \frac{\sigma(\xi - x)dS}{r^3}, \ldots, \ldots \right) = \iint_S \frac{\sigma \boldsymbol{r} dS}{r^3}, \tag{21}$$

and

$$U = \iint_S \frac{\sigma dS}{r}, \tag{22}$$

the density $\sigma$ being supposed bounded and integrable. This is the potential of a *single layer*, in contrast with what is called the *potential of a double layer* which will now be studied.

Electrical forces and magnetic forces obey the Coulomb law, which is exactly like the Newtonian law except that lik

charges *repel* instead of *attracting*. It is customary to use the same definition for the potential function for these fields, and to take care of the difference in sign of the field (from gravitational fields), by writing

$$F = -\operatorname{grad} U \tag{23}$$

for electric and magnetic fields.

Let $S$ be a surface free from singularities and one-sided, so that the normal direction is a continuous function of the position of a point on the surface. Select a positive direction for the normal, which is then defined by continuity over the whole surface. Let the surface $S$ carry an electrical charge of density $\sigma$, so that its potential function is $\iint \frac{\sigma dS}{r}$. In the negative normal direction from each point $Q$ of the surface locate $Q_1$ at a constant distance $h$, thus forming a parallel surface $S_1$. We assume that for sufficiently small $h$, the surface $S_1$ does not intersect itself; corresponding points $Q$ and $Q_1$ have the same normal. Let the surface $S_1$ carry a charge of density $\sigma_1$, such that corresponding area elements carry numerically equal charges of opposite signs, i.e.

$$\sigma_1 dS_1 = -\sigma dS.$$

Such a condition holds approximately on the two conducting coatings of a charged Leyden jar. The potential due to the two surfaces is then

$$U = \iint\limits_S \frac{\sigma dS}{r} + \iint\limits_{S_1} \frac{\sigma_1 dS_1}{r_1} = \iint\limits_S \sigma h \frac{\frac{1}{r} - \frac{1}{r_1}}{h} dS.$$

Now let $h \to 0$ and $\sigma \to \infty$, so that $\sigma h \to \mu$ everywhere uniformly on $S$; then also

$$\lim_{h \to 0} \frac{\frac{1}{r} - \frac{1}{r_1}}{h} = \frac{\partial}{\partial n}\left(\frac{1}{r}\right).$$

Hence the potential

$$U = \iint \mu \frac{\partial}{\partial n}\left(\frac{1}{r}\right) dS,$$

obtained as the limit of the potential of two single layers of opposite sign which approach coincidence, is called the potential of a double layer. The function $\mu$ is called the moment of the double layer, and is assumed to be bounded and integrable.

Since

$$\begin{aligned}\frac{\partial}{\partial n}\left(\frac{1}{r}\right) &= -\frac{1}{r^2}\frac{\partial r}{\partial n} \\ &= -\frac{1}{r^2}\left[\frac{\partial r}{\partial \xi}\cos(n,x) + \frac{\partial r}{\partial \eta}\cos(n,y) \right. \\ &\qquad\qquad \left. + \frac{\partial r}{\partial \zeta}\cos(n,z)\right] \\ &= -\frac{1}{r^2}\left[\frac{\xi - x}{r}\cos(n,x) + \dots\right] \\ &= -\frac{1}{r^2}\left[-\cos(r,x)\cos(n,x) - \dots\right] \\ &\qquad\qquad \text{(if } r \text{ is directed from } Q \text{ to } P\text{)} \\ &= \frac{1}{r^2}\cos(r,n),\end{aligned}$$

the potential of a double layer may be written

$$U = \iint \frac{\mu \cos(r,n)}{r^2}\, dS. \tag{24*}$$

(The potential of double layers was introduced into the theory by Helmholtz.)

For completeness, we remark that the Newtonian potential due to a distribution of mass along a curve, attracting by the

inverse square law, is $\int \frac{dm}{r} = \int \frac{\gamma ds}{r}$. We will make no use of this definition.

In the plane, consider a curve made of a finite number of pieces with continuously turning tangents; let this curve $C$ carry a distribution of density $\gamma$ acting under the inverse first power law. The logarithmic potential of this single curve distribution is then

$$U = \int \log \frac{1}{r}\, dm = \int \gamma \log \frac{1}{r}\, ds, \tag{25}$$

where $ds$ is the element of arc. By a passage to the limit exactly analogous to that used in defining a double layer, the expression

$$U = \int \gamma \frac{\partial}{\partial n}\left(\log \frac{1}{r}\right) ds = \int \frac{\gamma \cos (r, n)}{r}\, ds \tag{26}$$

is obtained as the potential of a double linear distribution. Also, the potentials of single and double layers are continuous and indefinitely often differentiable at all points outside the acting masses.

## Art. 5. The Laplace Equation

We will now study a very important property of the potential, which is common to all the potential functions which have been defined. We note that the function

$$\frac{1}{r} = \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}},$$

which may be considered as the potential (Newtonian) of a unit mass located at $(\xi, \eta, \zeta)$, satisfies a partial differential equation. For we find

$$\frac{\partial\left(\frac{1}{r}\right)}{\partial x} = \frac{\xi - x}{r^3}, \quad \frac{\partial^2\left(\frac{1}{r}\right)}{\partial x^2} = -\frac{1}{r^3} + \frac{3(\xi - x)^2}{r^5},$$

and similar equations for the derivatives with respect to $y$ and $z$. Hence

$$\frac{\partial^2\left(\frac{1}{r}\right)}{\partial x^2} + \frac{\partial^2\left(\frac{1}{r}\right)}{\partial y^2} + \frac{\partial^2\left(\frac{1}{r}\right)}{\partial z^2} = 0. \tag{27}$$

This is Laplace's equation, fundamental in the whole of potential theory. For convenience, we introduce the abbreviation

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}; \tag{28}$$

then the potential $\frac{m}{r}$ of a point-mass satisfies the equation

$$\nabla^2 U = 0. \tag{29}$$

Laplace's equation is a second order linear homogeneous partial differential equation of elliptic type. From its linear homogeneous character, it follows that if $u_1, u_2, u_3, \ldots, u_m$ are solutions, then any linear combination

$$c_1 u_1 + c_2 u_2 + \ldots + c_m u_m$$

with constant coefficients $c_i$ is a solution. Hence the potential $\Sigma \frac{m_i}{r_i}$ due to several point-masses satisfies (29) since each term of the sum does. Moreover, for the potential due to a body mass distribution, we have

$$\nabla^2 U = \iiint\limits_V \nabla^2\left(\frac{1}{r}\right) \rho dV = 0,$$

since differentiation under the integral sign is permissible, for all points $P{:}(x, y, z)$ outside the region $V$ where the distribution

is located. The same is true of the potential of a surface or single layer.

Likewise, the potential of a double layer satisfies Laplace's equation. To see this, it is merely necessary to note that the expression $\frac{\partial}{\partial n}\left(\frac{1}{r}\right)$, in which $\xi$, $\eta$, $\zeta$ are the variables for differentiation in the normal direction, satisfies Laplace's equation in $(x, y, z)$, since

$$\nabla^2\left(\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right)=\frac{\partial}{\partial n}\left(\nabla^2\left(\frac{1}{r}\right)\right)=0.$$

Correspondingly, in logarithmic potential, we find that

$$\frac{\partial}{\partial x}\left(\log\frac{1}{r}\right)=\frac{\xi-x}{r^2},\quad \frac{\partial^2}{\partial x^2}\left(\log\frac{1}{r}\right)=-\frac{1}{r^2}+\frac{2(\xi-x)^2}{r^4},$$

and hence

$$\text{0)}\qquad \frac{\partial^2}{\partial x^2}\left(\log\frac{1}{r}\right)+\frac{\partial^2}{\partial y^2}\left(\log\frac{1}{r}\right)=0.$$

Accordingly, for functions of two variables, we define

$$\text{(31)}\qquad \nabla^2 U=\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}.$$

Then the logarithmic potential of a point-mass, and accordingly all the forms of logarithmic potential, satisfy Laplace's equation in the plane, at all points outside the acting masses.

## Art. 6. Behaviour of the Potential at Infinity

We will next investigate the behaviour of the potential and its first derivatives when the field-point $P$ moves off to infinity. Consider the potential of a body

$$U=\iiint_V \frac{\rho\, dV}{r}.$$

Let $g$ and $G$ be the lower and upper limits of the distance $r$ from $P$ to the integration points $Q$ of the region $V$ (bounded); then for all points $Q$ in the volume $V$ we have

$$g \leq r \leq G$$

so that

$$\iiint \frac{\rho dV}{G} < \iiint \frac{\rho dV}{r} < \iiint \frac{\rho dV}{g},$$

or

$$\frac{M}{G} < U < \frac{M}{g}, \tag{32}$$

where $M = \iiint \rho dV$ is the total mass of the body. If now the point $P$ moves off to infinity, or $R \to \infty$ where $R$ is the distance from $P$ to the origin of coordinates, then $g \to \infty$ and $G \to \infty$, so that

$$\lim_{R \to} U = 0. \tag{33}$$

Hence the Newtonian potential vanishes at infinity. More exactly, since $\frac{G}{R} \to 1$ and $\frac{g}{R} \to 1$ as $R \to \infty$, and since

$$\frac{MR}{G} < RU < \frac{MR}{g},$$

we have

$$\lim_{R \to \infty} (RU) = M. \tag{34}$$

This proof assumes that the density $\rho$ is everywhere positive; but in the general case we may suppose that the positive portions of the mass and the negative portions respectively produce potentials $U_+$ and $U_-$, and show as above that $\lim (RU_+) \to M_+$ and $\lim (RU_-) \to M_-$ where $M_+$ and $M_-$ are the total amounts of positive and negative masses.

Then (34) follows by addition. In the same way it can be shown that the following inequalities for Newtonian potentials, and also the limit equations and inequalities for logarithmic potentials, are valid in the general case.

Moreover, since

$$\frac{\partial U}{\partial x} = \iiint \frac{1}{r^2} \frac{(\xi - x)}{r} \rho \, dV$$

and since $\dfrac{|\xi - x|}{r} \leq 1$, we have

$$\left|\frac{\partial U}{\partial x}\right| < \iiint \frac{\rho \, dV}{r^2} = J.$$

If we treat $J$ in the same manner that we treated $U$ above, we find

$$R^2 J \rightarrow M \text{ as } R \rightarrow \infty.$$

Hence $R^2 \dfrac{\partial U}{\partial x}$ for large values of $R$ has a value less than some bound $C$ which is independent of $R$. The same condition holds for all three first partial derivatives, which we will represent in general by $D_1 U$. We have therefore

$$R^2 |D_1 U| < C. \tag{35}$$

This condition implies the less sharp result that $D_1 U \rightarrow 0$ as $R \rightarrow \infty$. The relations (34) and (35) hold of course for the potential due to a single mass-point $m$ or to several masses $m_i$, and also for the case of a surface distribution of mass.

Finally, since the denominator in the integral (24*) for the potential of a double layer contains $r^2$, it follows as above that for a double layer $R^2 U$, instead of merely $RU$, is bounded. Hence for the potential of a double layer,

$$\lim_{R \rightarrow \infty} RU = 0. \tag{36}$$

This result can also be obtained from (34), since the total mass

of a double layer is zero.[2] Similarly, for a potential $U$ of a double layer, we find not merely $R^2|D_1U|$, but

$$R^3|D_1U| < C. \tag{35*}$$

This is stronger than (35), which is therefore valid *a fortiori*.

We may also obtain inequalities as follows: let

$$w = U - \frac{M}{R} = \iiint\limits_V \left(\frac{1}{r} - \frac{1}{R}\right) dm, \quad (dm = \rho dV),$$

then

$$R^2 w = \iiint (R - r)\frac{R}{r}\, dm.$$

But $|R - r|$ is less than the distance from $Q$ to the origin and is therefore bounded. Also $r/R \to 1$ as $R \to \infty$, so that this ratio is bounded, and is indeed uniformly bounded for $Q$ in $V$. Hence $R^2|w|$ is bounded. This is stronger than the equation $Rw \to 0$ which follows from $RU \to M$ as $R \to \infty$. Moreover,

$$\frac{\partial w}{\partial x} = \iiint \left(\frac{\xi - x}{r^3} + \frac{x}{R^3}\right) dm,$$

$$R^3 \frac{\partial w}{\partial x} = \iiint \xi \frac{R^3}{r^3}\, dm + x \iiint \frac{r^3 - R^3}{r^3}\, dm.$$

The first integral is bounded. The second is equal to

$$\frac{x}{R} \iiint \frac{r^3 - R^3}{R^2} \frac{R^3}{r^3}\, dm.$$

But $\left|\dfrac{x}{R}\right| \leq 1$ and $\dfrac{r^3 - R^3}{R^2} = (r - R)\dfrac{r^2 + rR + R^2}{R^2}$ is uniformly bounded. Hence the second integral and hence $R^3\left|\dfrac{\partial w}{\partial x}\right|$ is bounded, and hence also $R^3|D_1 w|$ is bounded. Similarly, it may be shown that, for the potential due to a surface distribution, $U = \displaystyle\iint \frac{dm}{r}$, the function $w = U - \dfrac{M}{R}$ has the

[2]This follows easily from the definition of the double layer in Art. 4.

same properties, namely that $R^2|w|$ and $R^3|D_1w|$ are bounded. For the potential $U$ of a double layer, which has the total mass zero, we have already proved that $R^2|U|$ and $R^3|D_1U|$ are bounded.

We turn now to the logarithmic potential, which is quite different from the Newtonian potential in its behaviour at infinity. For the potential $U = \iint \log \frac{1}{r}\, dm$, when the distance from $P$ to any point $Q$ of the mass is so great that $r > 1$ everywhere, we have

$$-U = |U| = \iint \log r\, dm.$$

Using the same method as above, we find

$$M \log g < |U| < M \log G,$$

or

$$\lim_{R \to \infty} U = -\infty, \tag{37}$$

which is very different from the result in (33). More exactly, since

$$\frac{\log g}{\log R} \to 1 \text{ and } \frac{\log G}{\log R} \to 1 \text{ as } R \to \infty \text{, we find } \frac{|U|}{\log R} \to M,$$

or

$$\lim_{R \to \infty} \frac{U}{\log \dfrac{1}{R}} = M. \tag{38}$$

We have

$$\frac{\partial U}{\partial x} = \iint \frac{1}{r} \cdot \frac{\xi - x}{r}\, dm$$

from which it follows easily that $R\left|\frac{\partial U}{\partial x}\right|$ is bounded; in general, for the derivative in any direction,

$$R|D_1U| < C. \tag{39}$$

The relations (38) and (39) hold also for the logarithmic potentials due to point-masses and masses distributed along lines.

For a double distribution (distribution of doublets) along a line, the total mass is again to be called zero. From the integral $U = \int \frac{\gamma \cos(r, n)}{r} ds$ of a double distribution along a line, since $r$ occurs in the denominator, it can be seen that $RU$ is bounded, and therefore $U \to 0$ as $R \to \infty$; this is stronger than $\frac{U}{\log R} \to 0$. For large values of $R$, not only $R|D_1 U|$ but also $R^2|D_1 U|$ is bounded.

Further relations can be obtained as follows. For the logarithmic potential $U$ of a distribution in the plane (inverse first power law), let

$$w(x, y) = U - M \log \frac{1}{R} = \iint \left( \log \frac{1}{r} - \log \frac{1}{R} \right) dm.$$

This can be written

$$w = -\iint_S \log \frac{r}{R}\, dm.$$

Since for all points $Q$ of $S$, $r/R \to 1$ or $\log \frac{r}{R} \to 0$ uniformly, it follows that $w \to 0$, as $R \to \infty$. It may even be easily proved that $Rw$ is bounded. Also, from the equation

$$\frac{\partial w}{\partial x} = \iint \left( \frac{\xi - x}{r^2} + \frac{x}{R^2} \right) dm,$$

it is easy to show that $R^2 \left| \frac{\partial w}{\partial x} \right|$, and in general $R^2 |D_1 w|$, is bounded. The logarithmic potential of line or point distributions has these properties also. For the potential (logar-

ithmic) of a double line distribution (mass $= 0$), it has already been shown that $U \to 0$ and $R^2|D_1U|$ is bounded as $R \to \infty$.

*Exercises*: Carry through the derivation of (35*), that $R^3|D_1U|$ remains bounded for the potential of a double layer, in detail from the definition $U = \iint \mu \dfrac{\partial\left(\frac{1}{r}\right)}{\partial n} dS$ by setting

$$\frac{\partial\left(\frac{1}{r}\right)}{\partial n} = -\frac{1}{r^2}\left\{\frac{\xi - x}{r}\cos(n, x) + \frac{(\eta - y)}{r}\cos(n, y) + \frac{(\zeta - z)}{r}\cos(n, z)\right\}$$

and carrying out the differentiation for $\dfrac{\partial U}{\partial x}$ under the integral sign.

Similarly carry out the proof that $R^2|D_1U|$ is bounded for the potential (logarithmic) of a double line distribution.

## Art. 7. Harmonic Functions. Regularity in the Finite Regions and at Infinity

As we have seen, every one of the potential functions which have been defined is continuous and indefinitely often differentiable at all finite points of free space (or plane),[3] and satisfies Laplace's equation. We will now take this equation as a starting point and define: *A function U which is continuous with continuous first and second derivatives in a finite region*[4] *G, and there satisfies*

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0, \text{ or } \nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0,$$

[3] that means: mass-free points.

[4] $G$ is an open connected point-set in space or in the plane.

*is called a regular harmonic function in G.* The function is also said to be regular in $G$. A harmonic function is said to be regular at a point if it is regular in some neighbourhood of this point. Such a point is called a "regular point" for the function. *The Newtonian and logarithmic potentials are regular harmonic functions at every finite mass-free point of space or the plane.* We will see later (Chapter 3, Art. 7) that every function regular and harmonic in a finite region may be represented as a potential function. The concepts are of "potential" and of "regular harmonic function" are therefore essentially equivalent.

A harmonic function (solution of Laplace's equation) will be called *regular at infinity* when it is regular in some neighbourhood of infinity, that is, for all finite points outside some sufficiently large sphere, and for $R = \sqrt{x^2 + y^2 + z^2} \to \infty$, satisfies the conditions

(40) $$\lim RU \text{ exists and } R^2 D_1 U \text{ is bounded.}$$

For example, the function $1/R$ is regular at infinity. The three Newtonian potential functions are regular at infinity.

The limit

(41) $$\lim_{R \to \infty} RU = M$$

is called the "*mass*" of the harmonic function. This purely analytic definition of the mass of any harmonic function which is regular at infinity is evidently in agreement with the concept of the total mass which produces a Newtonian potential, since (41) holds for all Newtonian potentials.

In the plane we make the following definition: A harmonic function $U(x, y)$ is said to be *regular at infinity* if it is regular outside some sufficiently large circle, and if

(42) $$\lim U = C \text{ exists and } R^2 D_1 U \text{ is bounded,}$$

when $R = \sqrt{x^2 + y^2} \to \infty$. We will later (Chapter 8, Art. 5)

find that the existence of lim $U$ alone ensures regularity at infinity. The conditions (42) are therefore not independent, though the conditions (40) were. Of the logarithmic potentials, the potential of a double distribution on a line is regular at infinity.

If for sufficiently large $R$,

$$U = M \log \frac{1}{R} + \tag{43}$$

where $M$ is a constant and $w$ is a harmonic function regular at infinity, then $M$ is called the "mass" of $U$. This analytic definition is in agreement with the total mass of a distribution causing a logarithmic potential. The logarithmic potential of a distribution in the plane is regular at infinity if and only if its total mass is zero, as is true for double distributions.

The mass of a Newtonian or logarithmic potential is independent of the coordinate-system, and so is the mass as defined by (41) or (43) respectively.

Since every harmonic function can be represented as a potential, and conversely every potential is a harmonic function, potential theory is also the theory of the Laplace equation. Since Laplace's equation is the simplest partial differential equation of elliptic type, potential theory serves as the foundation, the model and the introduction to the theory of elliptic partial differential equations. (The general linear partial differential equation of the second order

$$A_{11}\frac{\partial^2 u}{\partial x^2} + A_{22}\frac{\partial^2 u}{\partial y^2} + A_{33}\frac{\partial^2 u}{\partial z^2} + A_{12}\frac{\partial^2 u}{\partial x \partial y} + \ldots +$$

$$B_1 \frac{\partial u}{\partial x} + \ldots + Cu + D = 0$$

is *elliptic* if the quadratic form $\sum_{i,j=1}^{3} A_{ij}\, u_i\, u_j$ is "definite." Compare J. Hadamard, Cours d'analyse, Vol. II, p. 511. Paris 1930.)

**Art. 8. Equipotential Surfaces and Lines of Force**

Let $u$ be a Newtonian potential (harmonic function) which takes on the value $a$ at a mass-free (regular) point $P$, then the equation

$$u(x, y, z) = a \tag{44}$$

determines a surface passing through the point $P$. Such a surface is called an equipotential surface or level surface. It has in general a continuous normal and a continuous curvature, since the second partial derivatives of $u$ are continuous. An equipotential surface can only have a singular point where the three partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ vanish simultaneously, i.e. where grad $u = 0$. When $a$ is given different values, the equation (44) represents a one-parameter family of surfaces. Only one surface can pass through any point, since $u$ cannot have several different values in one point. Since $u$ is constant on a level surface, we get $\frac{\partial u}{\partial s} = 0$ for every tangential direction $s$. Therefore the force-vector (see Art. 1)

$$\boldsymbol{F} = \operatorname{grad} u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right. \tag{45}$$

is everywhere perpendicular to the surface. The lines of force are those curves which at each point have the same direction as the force there. They are accordingly perpendicular to the surfaces (44). They form the orthogonal trajectories to the surfaces $u$ = constant.

Correspondingly, in logarithmic potential, the lines of force are the orthogonal trajectories to the equipotential curves.

For the example of a point-mass located at $Q$: $(\xi, \eta, \zeta)$, the level or equipotential surfaces are the concentric spheres

$$r = \sqrt{(\xi - x)^2 + (y - \eta)^2 + (\zeta - z)^2} = \text{const.}$$

The lines of force are the straight lines passing through $Q$. Also in the case of the logarithmic potential of a point-mass, the equipotentials and lines of force are concentric circles and radial lines.

The lines of force have the differential equations (compare Chapter I, Art. 4)

$$dx : dy : dz = \frac{\partial u}{\partial x} : \frac{\partial u}{\partial y} : \frac{\partial u}{\partial z}, \tag{46}$$

where $dx$, $dy$, $dz$ represents a displacement along the line of force; or in the case of logarithmic potential

$$dx : dy = \frac{\partial u}{\partial x} : \frac{\partial u}{\partial y}. \tag{46*}$$

Consider the example $u = x^2 - y^2$, which evidently satisfies Laplace's equation. This potential is regular in the finite plane. The equipotentials form a set of equilateral hyperbolas. They have in general no singular points. The only exception is the degenerate hyperbola, which is formed by the pair of lines $y = x$, $y = -x$; on them the origin, where grad $u = 0$, is a singular point, and in this case a double point. The two branches of the degenerate hyperbola, i.e. the two lines, cut each other orthogonally there.

The lines of force are obtained from the differential equation $\frac{dy}{dx} = -\frac{y}{x}$ which leads to $xy = \text{const}$. They are likewise equilateral hyperbolas.

*Exercise*: Determine the lines of force for the equipotential surfaces $u = x^2 + y^2 - 2z^2 = \text{const}$.

## Art. 9. Examples and Problems

We will now apply the theorems derived in the preceding paragraphs to calculate the potentials of several mass distributions.

1. Since the potential of a given mass is defined by a definite integral, the potential in the various problems may be found by simplifying this integral and evaluating it when possible. But it will often prove to be easier to use the fact that each potential at points of free space is a solution of Laplace's equation. Some problems will show the value of the new method.

Consider the potential of a mass distribution lying between two concentric spheres, when the density is constant or merely a function of the distance from the centre of the spheres. On account of the symmetry, the potential must depend merely on the distance $R = \sqrt{x^2 + y^2 + z^2}$ from the centre, taken as the origin of coordinates. We therefore transform Laplace's equation, by setting $U = U(R)$. Then

$$\frac{\partial U}{\partial x} = \frac{dU}{dR} \cdot \frac{x}{R},$$

$$\frac{\partial^2 U}{\partial x^2} = \frac{d^2 U}{dR^2} \cdot \frac{x^2}{R^2} + \frac{dU}{dR}\left(\frac{1}{R} - \frac{x^2}{R^3}\right),$$

and corresponding expressions for the derivatives with respect to $y$ and $z$; hence

$$\nabla^2 U = \frac{d^2 U}{dR^2} + \frac{2}{R}\frac{dU}{dR} = 0.$$

Let $\dfrac{dU}{dR} = f(R)$ ; then $\dfrac{df}{dR} + \dfrac{2}{R} f = 0$, $\dfrac{df}{f} + \dfrac{2dR}{R} = 0$,

$$\log f + \log R^2 = \text{const.},\ fR^2 = -a,\ f = -\frac{a}{R^2},$$

and finally,
$$U = \frac{a}{R} + b,$$

where $a$ and $b$ are constants.

Now we must distinguish between the two cases, $P$ outside the outer sphere or inside the inner one. In the first

case, the condition (34) $RU \to M$ as $R \to \infty$ leads to the evaluation of the constants, $a = M$, $b = 0$, so that

$$U = \frac{M}{R}. \tag{47}$$

For the second case, since the potential must be regular at the origin, we have $a = 0$, so that

$$U = b. \tag{48}$$

The constant $b$ is determined by evaluating the integral for the potential at the origin (compare Chapter 5, Art. 10).

From (47) it is evident that the potential outside the larger sphere is the same as if the mass had all been concentrated at the centre. This is hence also true for the force

$$\boldsymbol{F} = \operatorname{grad} U = \left(-\frac{Mx}{R^3}, -\frac{My}{R^3}, -\frac{Mz}{R^3}\right).$$

Equation (48) shows that inside the inner sphere the force is zero, or the mass exerts no force at all on a point inside a hollow spherical shell.

By taking the radius of the inner sphere as zero, the above derivation gives the potential due to a complete spherical distribution, outside the sphere.

By passing to the other extreme, taking the radius of the inner sphere equal to that of the outer one, (47) and (48) also yield the potential outside and inside a spherical surface distribution of mass of constant density.

The corresponding problem in the plane is the problem of a distribution over the area between two concentric circles, attracting by the inverse first power law, when the density is a function of the distance from the centre only. Then $U = U(R)$, and Laplace's equation becomes

$$\nabla^2 U = \frac{d^2 U}{dR^2} + \frac{1}{R}\frac{dU}{dR} = 0.$$

Let $$\frac{dU}{dR} = f(R)\,, \quad \text{then } \frac{df}{dR} + \frac{f}{R} = 0\,, \; f = -\frac{a}{R}\,,$$

$$U = a \log \frac{1}{R} + b\,,$$

where $a$ and $b$ are constants. If $P$ is outside the outer circle, the behaviour at infinity, $U - M \log \frac{1}{R} \to 0$, leads to the evaluation of both constants, so that $a = M$, $b = 0$, and

$$U = M \log \frac{1}{R} \cdot$$

When $P$ is inside the inner circle, it is necessary that $a = 0$, so that

$$U = b.$$

*Remark.* In three-dimensional problems it may happen that the potential function is clearly independent of one variable, as in the field around infinite straight parallel wires or cylinders. Laplace's equation then reduces to the differential equation in two variables; hence logarithmic potential is important in electrical theory, where parallel conductors are often used.

2. Homogeneous straight wire $AB$ with mass of constant density $\gamma$ (see Chapter I, Art. 3).

Find the potential $U$ at an arbitrary point $P:(x, y, z)$. Take the wire as the $z$-axis. Let the coordinates of $A$ and $B$ be $a$ and $b$ respectively. Prove that

$$U = \int_a^b \frac{\gamma d\zeta}{\sqrt{x^2 + y^2 + (z - \zeta)^2}}\,,$$

$$= \gamma \log \frac{\sqrt{\rho^2 + (b - z)^2} + b - z}{\sqrt{\rho^2 + (a - z)^2} + a - z}\,,$$

$\rho = \sqrt{x^2 + y^2}$ being the distance of $P$ from the wire.

3. A constant force field is conservative. *Example*: The gravitational field of the earth in a sufficiently small range. Let $a$, $b$, $c$ be constants and let $X = a$, $Y = b$, $Z = c$. Find the potential, which is a linear function of $x$, $y$, $z$.

4. A force field is called "central" if the force-vector always passes through a fixed point, the "centre," and the magnitude of the force is a function of the distance $r$ from the centre only. *Example*: The Newtonian field due to a mass-point. A central force field is conservative. Let $f(r)$ be the magnitude of the force, so that $X = f(r)\dfrac{x}{r}$, $Y = f(r)\dfrac{y}{r}$, $Z = f(r)\dfrac{z}{r}\cdot$ Show that the equations (6) in Art. 1 are satisfied. Prove that $U = \int f(r)\,dr$ is the potential.

5. If the force vector is always perpendicular to a fixed line, the axis, and if the magnitude of the force is a function of the distance from this line only, the field is an "axial" field. *Example*: The infinite homogeneous straight wire, attracting by the Newtonian law (Chapter 1, Art. 3). An axial field is conservative. Let the axis be the $z$-axis. Introduce cylindrical coordinates $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z$. The magnitude of the force is then a function $f(\rho)$ of $\rho$. Determine the components of the force. Prove that $U = \int f(\rho)d\rho$ is the potential. In the case of the infinite wire we have $f(\rho) = \dfrac{2\gamma}{\rho}$. Show that $U = 2\gamma\log\dfrac{1}{\rho} + \text{constant}$.

A more general definition is the following: A field is axial if the force vector always passes through a fixed line and its magnitude depends on the distance from this line only. But such a field is not necessarily conservative; e.g., the field $\mathbf{F} = (x, y, f(\rho))$, where $\rho = \sqrt{x^2 + y^2}$, is axial in the above more general sense, but obviously not conservative.

# CHAPTER III

## THE INTEGRAL THEOREMS OF POTENTIAL THEORY

### Art. 1. Gauss's Theorem or Divergence Theorem

We will now obtain the divergence theorem, which is of fundamental importance in potential theory.

Let $V$ be a bounded, simply-connected and finite region of space, enclosed by a surface $S$ with continuously turning normal. Assume that any line parallel to the $x$-axis cuts $S$ in at most two points. Let $F_1$ and $\frac{\partial F_1}{\partial x}$ be continuous on $V + S$. Consider the integral

$$\iiint\limits_V \frac{\partial F_1}{\partial x} dV = \iint\left(\int \frac{\partial F_1}{\partial x} dx\right) dydz.$$

If we carry out the integration, with $y$ and $z$ constant, in the direction of the $x$-axis, this becomes (it is assumed $x_2 > x_1$)

$$\iint [F_1(x_2, y, z) - F_1(x_1, y, z)]\, dydz,$$

where the line $y = \text{const.}$, $z = \text{const.}$ cuts the surface $S$ in the points $P_1{:}(x_1, y, z)$ and $P_2{:}(x_2, y, z)$. The infinite rectangular cylinder erected on the area element $dydz$ cuts $S$ in the area elements $dS_1$ and $dS_2$ at $P_1$ and $P_2$ respectively. Let $\boldsymbol{n}$ be the exterior or outward-pointing unit normal to the surface; this makes an acute angle with the $x$-axis at $P_2$ and an obtuse angle at $P_1$, so that we have (Fig. 4)

$$dydz = -dS_1 \cos(n_1, x), \quad dydz = dS_2 \cos(n_2, x).$$

The integral then becomes

$$\iint [F_1(x_2, y, z) \cos(n_2, x) dS_2 + F_1(x_1, y, z) \cos(n_1, x) dS_1]$$

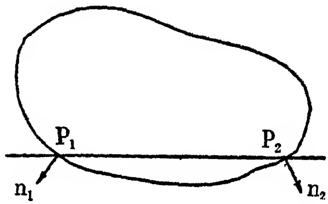


FIG. 4

which reduces to a surface integral; the final result may then be written

$$\iiint\limits_V \frac{\partial F_1}{\partial x} dV = \iint\limits_S F_1 \cos(n, x) dS,$$

where the integral on the right is taken over the complete surface $S$. This formula was obtained by Gauss. It is of importance that it transforms a volume integral into a surface integral.

The hypotheses of the above theorem may be made much less restrictive. If the surface is such that a parallel to the $x$-axis can cut it in a finite number of points $P_k(k = 1, 2, 3, \ldots)$, the number of points will be even since the surface is closed, and the outward normal will make alternately obtuse and acute angles with the $x$-direction at the successive points. The equation (1) will still hold. Also the region $V$ need not be simply-connected; it may be multiply-connected, or it may be composed of several unconnected regions $V_i$ each enclosed by the corresponding surface $S_i(i = 1, 2, \ldots)$. The surface integral is then to be the sum of the integrals over all the bounding surfaces,

$$\iint\limits_S = \iint\limits_{S_1} + \iint\limits_{S_2} + \cdots$$

Of course the theorem also holds for regions such as the space between two concentric spheres; here outward normal means the normal to the surface pointing away from the volume $V$, i.e. outward from the outer surface and inward from the inner surface, and the surface integral is taken over the two bounding surfaces.

The bounding surface $S$ may contain a finite number of edges, which divide the surface into a finite number of pieces with continuously changing normal direction. Also a finite number of singular points on $S$ are allowable, as for example the vertex of a cone. Such a point may at first be excluded, by truncating the cone by a plane; the theorem is valid for the truncated region, and remains valid in the limit as the plane moves to the vertex of the cone.

In a similar manner, we may obtain the equations

$$\iiint\limits_V \frac{\partial F_2}{\partial y} dV = \iint\limits_S F_2 \cos(n, y) dS,$$

$$\iiint\limits_V \frac{\partial F_3}{\partial z} dV = \iint\limits_S F_3 \cos(n, z) dS.$$

By addition of these three equations, we obtain *Gauss's theorem* or *the divergence theorem*

(2) $$\iiint\limits_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dV =$$

$$\iint\limits_S (F_1 \cos(n, x) + F_2 \cos(n, y) + F_3 \cos(n, z)) dS.$$

where $F_1$, $F_2$, $F_3$ are any three functions continuous in the closed region $V + S$ with continuous partial derivatives there, $\boldsymbol{n}$ is the outward normal, and the surface (or surfaces) $S$ is as described above. By defining the vector $\boldsymbol{F} = (F_1, F_2, F_3)$, this theorem may be written

$$\iiint\limits_V \nabla \cdot \boldsymbol{F} dV = \iint\limits_S \boldsymbol{n} \cdot \boldsymbol{F} dS. \tag{2*}$$

In the physical application where $\boldsymbol{F}$ is the velocity of flow of a fluid (see Art. 2) it is easily seen that the integral on the right represents the flow outward through $S$ in cubic units per unit time. This accounts for the name *divergence theorem*, and the name *divergence* for the quantity

$$\operatorname{div} \boldsymbol{F} = \nabla \cdot \boldsymbol{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Consider now a plane region $S$ bounded by the curve $C$ (or curves), which is such that $C$ is cut by a parallel to either axis only a finite number of times, and $C$ is composed of a finite number of pieces having a continuously turning tangent. Let $F_1$, $F_2$, $\frac{\partial F_1}{\partial x}$ and $\frac{\partial F_2}{\partial y}$ be continuous in the closed region $S + C$. Then we may prove in a similar way that

$$\iint\limits_S \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) dS = \int\limits_C (F_1 \cos(n, x) + F_2 \cos(n, y)) ds \tag{3}$$

or

$$\iint\limits_S \nabla \cdot \boldsymbol{F} dS = \int\limits_C \boldsymbol{n} \cdot \boldsymbol{F} ds, \tag{3*}$$

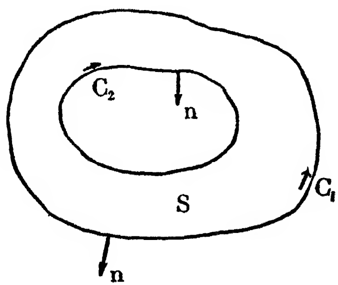


FIG. 5

which may also be written in the form

$$(3^{**})\qquad \iint\limits_S \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) dS = \int\limits_C (F_1 dy - F_2 dx).$$

In this last form, the line integral is to be taken in such a direction that the region $S$ lies along the left side of $C$. The region $S$ may be multiply-connected or may consist of several unconnected regions $S_i$ bounded by curves $C_i$. (Fig. 5.)

*Exercise.* By using (3) or (3**), express the area of a region in terms of a line integral around its boundary.

## Art. 2. Divergence. Solenoidal Fields

To substantiate the statement made above about fluid flow, it is merely necessary to note that the fluid which flows in unit time through the element of surface $dS$ would fill a tube with base area $dS$ and altitude $F_n$ where $F_n$ is the normal component of the velocity $\boldsymbol{F}$. Counting flow inward as negative outflow, it is then evident that

$$(4)\qquad \iiint\limits_V \operatorname{div} \boldsymbol{F} dV = \iint\limits_S \boldsymbol{n} \cdot \boldsymbol{F} dS = \iint\limits_S F_n dS$$

is the net outflow in cubic units per second. From (2*), by using the theorem of the mean, it is evident that the net outflow per unit time is

$$\iiint\limits_V \operatorname{div} \boldsymbol{F} dV = (\operatorname{div} \boldsymbol{F})_m \iiint\limits_V dv = V \cdot (\operatorname{div} \boldsymbol{F})_m,$$

where $(\operatorname{div} \boldsymbol{F})_m$ is a mean value of the quantity

$$\operatorname{div} \boldsymbol{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

over the region $V$, or is the value of this quantity at some point inside $V$ since $\operatorname{div} \boldsymbol{F}$ is by hypothesis a continuous function.

By letting $V$ shrink toward a point $P$, it may be seen that the divergence of the field $\boldsymbol{F}$ at $P$ may be defined as

$$\text{(5)} \qquad \operatorname{div} \boldsymbol{F} = \lim_{V \to 0} \frac{\displaystyle\iint_S \boldsymbol{n} \cdot \boldsymbol{F} dS}{V}.$$

These considerations make it evident that the quantity div $\boldsymbol{F}$ and the two integrals in (2) are independent of the coordinate system. The integral on the left in (4) is called "total divergence" for the region $V$.

If the fluid flowing is an incompressible liquid, and there are no sources or sinks (points where liquid is produced or destroyed), then the net outflow from any such region of space must be zero. It follows that the divergence of the velocity must be zero at each point in the region. Conversely, if the divergence vanishes everywhere, the net outflow from any region is zero and the liquid is incompressible. Hence

$$\text{(6)} \qquad \operatorname{div} \boldsymbol{F} = 0$$

is called the *equation of continuity* for the flow of velocity $\boldsymbol{F}$ of an incompressible liquid, and expresses the condition of incompressibility.

From the divergence theorem, it is seen that the condition

$$\operatorname{div} \boldsymbol{F} = 0$$

is equivalent to the condition

$$\text{(7)} \qquad \iint F_n dS = \iint \boldsymbol{n} \cdot \boldsymbol{F} dS = 0$$

for every closed surface $S$.

If the field $\boldsymbol{F}$ has a potential, and hence is the gradient of a scalar function $U$, then

$$\text{(8)} \qquad \operatorname{div} \boldsymbol{F} = \operatorname{div} \operatorname{grad} U = \nabla^2 U,$$

so that the divergence of $\boldsymbol{F}$ is the Laplacian of the potential

function $U$. If $U$ is any scalar point-function, as for example the temperature at the points inside any object, then

$$\text{div grad } U = \nabla^2 U \tag{9}$$

is likewise independent of the coordinate system.

When $\boldsymbol{F}$ has a potential function $U$, the divergence theorem takes the form

$$\iiint\limits_V \nabla^2 U dV = \iint\limits_S \frac{\partial U}{\partial n} dS. \tag{10}$$

The condition for incompressibility in flow of a fluid with velocity $\boldsymbol{F}$ becomes Laplace's equation

$$\nabla^2 U = 0, \tag{11}$$

or the condition

$$\iint\limits_S \frac{\partial U}{\partial n} dS = 0 \tag{12}$$

for every closed surface $S$ in the field.

If the given field is a force field, the integral

$$\iint\limits_S F_n dS \tag{13}$$

is called the flux of force outward through the closed surface $S$. The divergence theorem therefore states: *the total divergence for a region $V$ is equal to the flux of the force outward through the bounding surface $S$.* A force field which has zero divergence throughout a region is called a *solenoidal field* in this region. This term has therefore the same meaning as the term "incompressible" in connection with fluid flow. The flux of force out from a closed surface is zero, provided that the surface bounds a region throughout which the field is solenoidal.

*Newtonian fields are solenoidal at all points outside the actual mass distribution*, from equations (11) and (8). Newtonian fields are therefore both conservative and solenoidal. Con-

versely, if a force field is both conservative and solenoidal, then it has a potential $U$ which satisfies Laplace's equation and is therefore a harmonic function. (We will see in Art. 7 that every harmonic function can be represented as a Newtonian potential. Hence every conservative, solenoidal force field may be considered as a Newtonian force field.)

Consider all the lines of force which pass through the points of a given small closed curve. In general they form a tube-shaped surface which is called a tube of force. Let such a tube of force be cut by surfaces perpendicular to the lines of force in two places. This gives a region $V$ bounded by the wall of the tube and the two cross-sections $S_1$, $S_2$. The flux of force through the wall of the tube is zero, since the normal component $F_n$ is zero at each point of the wall (the force has the same direction as the line of force at each point). If the field is solenoidal, the flux of force out from the entire surface of $V$ is zero, so that

$$\iint_{S_1} F_n dS + \iint_{S_2} F_n dS = 0,$$

where the outer normal is used on $S_1$ and $S_2$. If we reverse the normal on $S_1$, this equation becomes

$$\iint_{S_1} F_n dS = \iint_{S_2} F_n dS.$$

From this we have approximately

$$S_1(F)_1 = S_2(F)_2$$

where $S_1$ and $S_2$ represent the cross-section areas of the tube at two places, and $(F)_1$, $(F)_2$ are the magnitudes of the force at the two places. Hence it is seen that for solenoidal fields the strength of the field at various points along a small tube of force is approximately inversely proportional to the cross-section area of the tube. The smaller the cross-section of the

tube, or the closer the lines of force crowd together, the stronger is the field. The more the lines separate, the weaker the field. The lines of force for solenoidal fields (Newtonian, for example) therefore give information about the strength of the field as well as its direction. The term "solenoidal" comes from the Greek word σωλην (the tube).

### Art. 3. The Flux of Force through a Closed Surface

A Newtonian field is solenoidal at all points in "free" space, that is, outside the attracting masses. Therefore the flux outward vanishes in a Newtonian field for all closed surfaces $S$ which contain no masses. On the other hand, this is no longer true when the surface $S$ contains masses in its interior. We will investigate this case.

First consider a surface $S$ containing a point-mass $m$ at $Q$ in its interior. Let $T$ be a small spherical surface about $Q$ of radius $r$, small enough to lie entirely within $S$. Since $U = m/r$,

$$(14)\quad \iint_T F_n dS = \iint_T \frac{\partial U}{\partial n} dS = \iint_T -\frac{m}{r^2} dS = -\frac{m}{r^2}\cdot 4\pi r^2 = -4\pi m,$$

where $n$ is the outward normal to the sphere $T$. In the region $V'$ between $T$ and $S$, the field is solenoidal, so that

$$(15)\qquad \iint_S \frac{\partial U}{\partial n} dS + \iint_T \frac{\partial U}{\partial n} dS = 0,$$

where the normal is outward on $S$, but inward on $T$. Combining these equations and noticing the reversal of the normal on $T$, we find

$$(16)\qquad \iint_S F_n dS = \iint_S \frac{\partial U}{\partial n} dS = -4\pi m,$$

where the normal is directed outward. Now let $V$ contain a distributed mass of total magnitude $M$ and density $\rho$; let this

distributed mass be contained in a region $V_1$ lying entirely interior to $V$. Then $U = \iiint\limits_{V_1} \frac{\rho}{r}\, dV$, where $r$ is the distance from the element of mass at $Q:(\xi,\ \eta,\ \zeta)$ to the field point $P:(x, y, z)$. Hence

$$\iint\limits_S \frac{\partial U}{\partial n}\, dS = \iint\limits_S \left\{ \iiint\limits_{V_1} \rho \frac{\partial \left(\frac{1}{r}\right)}{\partial n}\, dV \right\} dS.$$

Since $Q$ varies in $V_1$ and $P$ varies on $S$, the integrand is finite so that the integral is a proper one and the order of integration may be changed, giving

$$\iint\limits_S \frac{\partial U}{\partial n}\, dS = \iiint\limits_{V_1} \rho \left\{ \iint\limits_S \frac{\partial \left(\frac{1}{r}\right)}{\partial n}\, dS \right\} dV.$$

But from the consideration of a point-mass above,

$$\iint\limits_S \frac{\partial}{\partial n}\left(\frac{1}{r}\right) dS = -4\pi .$$

(This is the flux through $S$ of the force due to a unit mass at $Q$. See also Art. 7 of this chapter.) Hence

$$\iint\limits_S \frac{\partial U}{\partial n}\, dS = -4\pi \iiint\limits_{V_1} \rho\, dV = -4\pi M. \tag{17}$$

It is immaterial whether or not there are also present other masses outside $S$, because the field due to such masses is solenoidal inside $S$ and therefore gives no outward flux from $S$. It is also immaterial if the mass distribution reaches out to the surface $S$ instead of lying entirely inside it. In this case let $S$ be a slightly larger surface containing $S$ and hence $V$ in its interior; then the flux outward through $S$ is $-4\pi M$. But the field is solenoidal in the space between $S$ and $\bar{S}$; by letting $\bar{S}$

deform continuously into $S$, it follows that the theorem (17) holds for $S$. The theorem (also due to Gauss) may be stated: *the flux of force through a closed surface is $-4\pi M$, where $M$ is the mass contained in $S$.* Of course, the mass is supposed to be acting according to Newton's law.

**Remark:** The potential $U$ satisfies (17), and more( according to the divergence theorem

$$\iint \frac{\partial U}{\partial n} dS = \iiint \nabla^2 U dV,$$

therefore

$$\iiint \nabla^2 U dV = -4\pi M.$$

It is supposed that $\nabla^2 U$ is continuous in the region of integration. We apply the above equation to the neighbourhood $V$ of a point $P$ (inside the masses) and obtain

$$V \cdot \overline{\nabla^2 U} = -4\pi m \quad \text{or} \quad \overline{\nabla^2 U} = -4\pi \frac{m}{V},$$

$m$ being the mass of the volume $V$ and $\overline{\nabla^2 U}$ a mean value of $\nabla^2 U$ in $V$.

When $V \to 0$, containing the point $P$, and $\lim \frac{m}{V} = \tau$, the density in $P$, it finally follows that

$$\nabla^2 U = -4\pi\tau.$$

This is Poisson's equation. It holds if $\nabla^2 U$ is continuous in some neighbourhood of $P$. In Chapter V, Article 3, we will see that the condition of continuity is satisfied if $\tau$ is continuous with continuous derivatives of the first order.

### Art. 4. Stokes' Theorem

Stokes' theorem is, from a mathematical point of view, a simple application to a curved surface of the divergence theorem in the plane.

Let $S$ be an open curved surface, bounded by a closed curve $C$. Let $S$ have a continuously changing normal, and be cut by any straight line, and hence by any parallel to a coordinate axis, only a finite number of times. Also let $C$ be a closed curve without double points and having in general a continuously turning tangent. Choose arbitrarily a positive direction for the normal of $S$, and then choose a corresponding positive direction for the boundary $C$, such that the positive direction around $C$ is counter-clockwise when viewed from the positive side of $S$. Also assume that the surface $S$ may be represented by a function of the form

$$z = f(x, y),$$

where $f(x, y)$ is defined on the projection $T$ of $S$ on the $(x, y)$-plane and has continuous first partial derivatives there. (It has already been assumed to have a continuous normal.) Let $D$ be the boundary of $T$, or the projection of $C$. The direction cosines or components of the unit normal $\boldsymbol{n}$ satisfy the relation

$$\cos(n, x) : \cos(n, y) : \cos(n, z) = \frac{\partial f}{\partial x} : \frac{\partial f}{\partial y} : -1.$$

Now let $X(x, y, z)$ be defined and continuous with continuous first partial derivatives in a region of space containing $S$. On $S$ this can be represented as a function of $(x, y)$,

$$X = X(x, y, f) = g(x, y),$$

which is continuous with continuous partial derivatives in $T$. From the divergence theorem in the plane, which is valid under the present hypotheses,

$$-\iint_T \frac{\partial g}{\partial y}\, dxdy = \oint_D g\,dx,$$

where the integral around $D$ is in the positive direction in the plane. Now $\oint_D g\,dx = \oint_C X\,dx$ because the value of $X$ on $C$ is the

same as that of $g$ on $D$, and also

$$\frac{\partial g}{\partial y} = \frac{\partial X}{\partial y} + \frac{\partial X}{\partial z}\frac{\partial f}{\partial y} = \frac{\partial X}{\partial y} - \frac{\partial X}{\partial z}\frac{\cos(n, y)}{\cos(n, z)}$$

and $dxdy = \cos(n, z)dS$, since $dxdy$ is the projection of $dS$ on $T$. If we therefore transfer the integration from $T$ to $S$, we get

$$(18) \quad \iint\limits_S \left(\frac{\partial X}{\partial z}\cos(n, y) - \frac{\partial X}{\partial y}\cos(n, z)\right)dS = \oint\limits_C Xdx.$$

By cyclic permutation, we obtain two similar formulas, where $Y$, $Z$ are assumed to be continuous with first partial derivatives. By adding these formulas, we get *Stokes' theorem*,

$$(19) \quad \iint\limits_S \left[\left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\right)\cos(n, x) + \left(\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}\right)\cos(n, y) + \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right)\cos(n, z)\right]dS = \oint\limits_C (Xdx + Ydy + Zdz)$$

We can relax some of the restrictive hypotheses on the surface $S$. It is sufficient to suppose $S$ to be composed of a finite number of pieces of the kind described above, which join along continuous curves with in general continuously turning tangents. The formula (19) is valid for each such piece of $S$. By the addition of these formulas, the line integrals along the common edges between neighbouring areas cancel since these lines are traversed once in each direction, and the formula (19) for the entire area $S$ results.

**Art. 5. Curl**

If $X$, $Y$, $Z$ are the components of a vector $\boldsymbol{F}$,

$$\oint (Xdx + Ydy + Zdz) = \oint \boldsymbol{F}\cdot d\boldsymbol{r}$$

is the integral already discussed in Chapter II, Art. 1. The

right side of (19) is accordingly independent of the coordinate system, so that the left side must also have a meaning independent of the coordinate system. In fact, we shall find that the integrand $\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\Big)\imath + .$ of the left side of (19) is the normal component of a vector.

Let P : $(x, y, z)$ be an arbitrary point of the field and $\boldsymbol{n}$ an arbitrary direction there. Let $S$ be a small piece of surface through $P$ perpendicular to $\boldsymbol{n}$, and bounded by the curve $C$, satisfying the hypotheses of Stokes' theorem. Then the theorem holds for this surface and curve, and by using the mean value theorem, we find that

$$(20) \quad \left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\right)\cos(n, x) + \left(\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}\right)\cos(n, y) +$$

$$\left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right)\cos(n, z) = \lim \quad \boldsymbol{F} \cdot d\boldsymbol{r}$$

where $S$ has also been used to represent the area of the surface $S$. The left member of (20) is independent of the shape of $S$ and $C$, hence the right member must be also. Again, the right member is independent of the choice of coordinate system, so the left member is also. This expression is therefore dependent only on the field $\boldsymbol{F}$ and the direction $\boldsymbol{n}$. It is, from its form, the component of a vector in the direction of the unit vector $\boldsymbol{n}$. This vector is called the curl of the vector field $\boldsymbol{F}$, and is written

$$(21) \quad \text{curl } \boldsymbol{F} = \nabla \times \boldsymbol{F} = \left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z},\ \frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x},\ \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right).$$

Much simpler is the following proof:

By the expression $\left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\right)\cos(n, x) + \ldots$ a number is "associated" with any direction $\boldsymbol{n}$, the coefficients $\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}$,

etc. of $\cos(n, x)$, $\cos(n, y)$, $\cos(n, z)$ being independent of $\boldsymbol{n}$. Therefore a vector is defined (see introduction, abstract definition of a vector), having the components $\dfrac{\partial Z}{\partial y} - \dfrac{\partial Y}{\partial z}, \ldots$ That is the curl.

Stokes' theorem can therefore be written

$$\iint \operatorname{curl}_n \boldsymbol{F} dS = \oint_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} \tag{22}$$

which is independent of the coordinate system. Here $S$ is an open arbitrary (but necessarily two-sided) surface bounded by the curve $C$, and $\operatorname{curl}_n \boldsymbol{F}$ is the normal component of curl $\boldsymbol{F}$. The line integral around $C$ must be taken in the proper direction, counter-clockwise when viewed from the positive end of the normal.

If the field has the property

$$\operatorname{curl} \boldsymbol{F} \equiv 0, \tag{23}$$

then it follows from (22) that

$$\oint_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = 0$$

for all closed curves in the field. Such a field has a potential (Chapter II, Art. 1). The vanishing of the curl everywhere in a region is therefore a sufficient as well as a necessary condition for the existence of a potential (compare Chapter II, Art. 1).

In hydrodynamics, where $\boldsymbol{v}$ is the velocity of the fluid, the integral

$$\int \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{r}$$

is known as the "circulation" of the fluid along the curve $C$.

If this integral does not vanish when taken around a closed curve, the fluid is said to contain a vortex. In this case, curl $\boldsymbol{v}$ is not everywhere zero. The name "curl" signifies a rotation or vortex. When the circulation vanishes around every closed curve, or curl $\boldsymbol{v} \equiv 0$, the motion of the fluid is said to be vortex-free. In this case a "velocity potential" exists, which has the velocity as its gradient. "Vortex-free" fields in hydrodynamics correspond to "conservative" force fields.

## Art. 6. Green's Formulas

Let the region $V$ bounded by the surface $S$ satisfy the same hypotheses as in Art. 1. Let $u(x, y, z)$ and $v(x, y, z)$ be two continuous functions with continuous first and second derivatives in the closed region $V + S$. In the divergence theorem, let $\boldsymbol{F} = u$ grad $v$, and since

$$\text{(24)} \qquad \frac{\partial v}{\partial n} = \left(\frac{\partial v}{\partial x} n_1 + \frac{\partial v}{\partial y} n_2 + \frac{\partial v}{\partial z} n_3\right) = \boldsymbol{n} \cdot \text{grad } v,$$

we get the theorem

$$\text{(25)} \qquad \iiint\limits_V u \nabla^2 v \, dV + \iiint\limits_V \nabla u \cdot \nabla v \, dV = \iint\limits_S u \frac{\partial v}{\partial n} \, dS.$$

By interchanging $u$ and $v$, a similar formula is obtained, and by subtraction from (25), we find finally

$$\text{(26)} \qquad \iiint\limits_V (u \nabla^2 v - v \nabla^2 u) \, dV = \iint\limits_S \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}\right) dS.$$

The identities (25) and (26) are known as Green's formulas. The latter is more frequently useful.

Since only the first derivatives of $u$ amd $v$ enter in the surface integrals, and these only in the direction of the normal, the hypotheses mav be somewhat lightened. It is sufficient if $u$ and $v$ have continuous second derivatives merely in the

interior of the region $V$, while $u$, $v$, $\frac{\partial u}{\partial n}$ and $\frac{\partial v}{\partial n}$ remain continuous in the closed region $V + S$. To prove this, construct a parallel surface $S_1$ just inside $S$ at a distance $h$ from it, so that $S_1$ lies in $V$. (We now assume that $S$ is free from singularities.) For small enough $h$, it is assumed that $S_1$ does not cut itself, and hence encloses a volume $V_1$. Since $V_1 + S_1$ lies entirely within $V$, we can apply Green's formulas to it and obtain, for example,

$$\iiint\limits_{V_1} (u\nabla^2 v - v\nabla^2 u)dV = \iint\limits_{S_1} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dS.$$

Now let the distance $h$ between the parallel surfaces approach zero; then from the assumed continuity properties of $u$, $v$ $\frac{\partial u}{\partial n}$ and $\frac{\partial v}{\partial n}$, we obtain Green's theorem (26).

Furthermore, it is sufficient to assume that the second derivatives of $u$ and $v$ are merely piecewise continuous in $V$. Then Green's formulas are valid for each of the subregions into which the surfaces of discontinuity divide $V$. Then by addition of these formulas for the subregions, we obtain the theorems for the entire region; for because of the continuity of the functions and their first derivatives, the surface integrals cancel out over the interior dividing surfaces.

We will obtain some important conclusions from the Green's formulas by specializing the functions $u$ and $v$.

By letting $v = 1$ in (26), we get

$$\iiint\limits_V \nabla^2 u\, dV = \iint\limits_S \frac{\partial u}{\partial n}dS. \tag{27}$$

By letting $u = v$ in (25), we get

$$\iiint\limits_V (\nabla u)^2 dV = \iint\limits_S u\frac{\partial u}{\partial n}dS - \iiint\limits_V u\nabla^2 u\, dV. \tag{28}$$

If $u$ is a regular harmonic function in $V$, then since $\nabla^2 u = 0$, these equations become

$$\iint_S \frac{\partial u}{\partial n}\, dS = 0 \tag{29}$$

and

$$\iiint_V (\nabla u)^2 dV = \iint_S u \frac{\partial u}{\partial n}\, dS. \tag{30}$$

If $u$ and $v$ are both harmonic functions inside the closed surface $S$, then

$$\iint_S \left( u \frac{\partial v}{\partial n} \qquad \partial n \right) \tag{31}$$

We can extend the region of validity of equations (30) and (31) for harmonic functions, by dropping the restriction that the region $V$ be finite. Suppose $V$ to be an infinite region, bounded by one or more closed surfaces $S$, so that $V$ is the region exterior to the surfaces. Infinite regions with infinite boundaries, as for example the space between two parallel planes, are not considered here. The formulas (31) and (30) remain valid for the infinite regions $V$ described, if $u$ and $v$ are regular at infinity; the surface integral is taken over the surfaces $S$ which form the entire boundary. To prove this, let $F$ be a spherical surface of radius $R$ about the origin, where this radius is so large that the sphere $F$ contains all the boundary $S$ in its interior. In the region $V_1$ bounded by $S$ and $F$, $u$ is regular, and hence from (30),

$$\iiint_{V_1} (\nabla u)^2 dV = \iint_S u \frac{\partial u}{\partial n}\, dS + \iint_F u \frac{\partial u}{\partial n}\, dS.$$

By introducing polar coordinates (spherical coordinates) on $F$,

we have

$$\iint_F u \frac{\partial u}{dn} dS = \int_0^{\pi} \int_0^{2\pi} u \frac{\partial u}{dn} R^2 \sin \theta d\theta d\phi.$$

Now on account of the regularity of $u$ at infinity, $u \to 0$ and indeed $Ru$ and $R^2 \dfrac{\partial u}{\partial n}$ are bounded as $R \to \infty$. Hence it follows that the integral over $F$ has the limit 0 as $R \to \infty$. Similarly, the validity of the formula (31) for the infinite region $V$ can be proved.

On the other hand, the equation (29) is in general not valid for an infinite region $V$. It was derived by setting $v = 1$ in Green's theorem, and the function 1 is not regular at infinity. Let $G$ be any large closed surface which contains the entire boundary $S$ of $V$ in its interior, and let $V_1$ be the space between $G$ and $S$. The function $u$ is regular in this region, and hence from (29),

$$\iint_S \frac{\partial u}{\partial n} dS + \iint_G \frac{\partial u}{\partial n} dS = 0,$$

where the normals on both $S$ and $G$ are pointing away from the region $V$. By reversing the direction of the normal on $S$, this equation becomes

$$\frac{\partial u}{\partial n} dS = \iint_G \frac{\partial u}{\partial n} dS.$$

Therefore, if the harmonic function $u$ is regular in the infinite region $V$, then $\iint_G \dfrac{\partial u}{\partial n} dS$ has the same value for all closed surfaces $G$ which contain $S$ in their interior. The value of this integral is $-4\pi M$, where $M$ is the limit of $Ru$ as $R \to \infty$ (see Chapter II, Art. 7, eqn. (40)). This will be proved in the next paragraph.

Let $u$ be a potential, whose total mass $M$ lies inside the surface or surfaces $S$; then $u$ is regular in the region $V$ outside $S$, and it follows from Art. 3 that the above integral has the value $-4\pi M$ as stated. But we will prove only in the next Article that every harmonic function is a potential.

In the two-dimensional (logarithmic) case, the corresponding formulas are valid and are proved in a completely similar manner. We designate the formulas corresponding to (25), (26), . . . , (31) by starred numbers, as (25*), etc. Thus, for example,

$$(26^*) \qquad \iint\limits_S (u\nabla^2 v - v\nabla^2 u)dS = \oint\limits_C \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)ds,$$

$$(29^*) \qquad \oint\limits_C \frac{\partial u}{\partial n}ds = 0, \text{ etc.}$$

Also the extension of (30*) and (31*) to infinite regions in the plane can be easily justified. However, in the plane the formula (29*) is also valid for an infinite region; the function $v = 1$ is here regular at infinity.

*Exercise*: Verify the formula (29*) by evaluating the integral, when $u = x^2 - y^2$ and $C$ is the unit circle.

## Art. 7. Representation of a Harmonic Function as a Potential

In Green's formula (26), let $v = 1/r$, where

$$r = \sqrt{(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2},$$

taking $(\xi, \eta, \zeta)$ as integration variables and $(x, y, z)$ as parameters. Consider first the case where $P$ lies outside $V$. Then $v$ is a regular harmonic function in $V$, so that $\nabla^2 v = 0$, and we get

$$(32) \qquad \iiint\limits_V \frac{1}{r}\nabla^2 u\,dV = \iint\limits_S \left\{\frac{1}{r}\frac{\partial u}{\partial n} - u\frac{\partial\left(\frac{1}{r}\right)}{\partial n}\right\}dS.$$

If $u$ is regular harmonic in $V$, this reduces to

$$\iint_S \left\{ \frac{1}{r} \frac{\partial u}{\partial n} - u \frac{\partial \left( \frac{1}{r} \right)}{\partial n} \right\} dS = 0. \tag{33}$$

When $P$ is in the region $V$, the application of Green's theorem must be made in a different manner. Since $P:(x, y, z)$ is a singular point for $v$, it must be excluded from the region of integration. We therefore surround $P$ by a small sphere $K$ of radius $h$ and apply (26) to the region $V^*$ bounded by $S$ and $K$. Afterwards we let $h \to 0$. We obtain

$$\iiint_V \frac{1}{r} \nabla^2 u \, dV = \iint_S \left\{ \frac{1}{r} \frac{\partial u}{\partial n} - u \frac{\partial \left( \frac{1}{r} \right)}{\partial n} \right\} dS + \iint_K \left\{ \frac{1}{r} \frac{\partial u}{\partial n} - u \frac{\partial \left( \frac{1}{r} \right)}{\partial n} \right\} dS.$$

On $K$ we have $r = \text{const.} = h$, and $\frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$ since the normal is directed outward from $V^*$ and hence inward on $K$. By introducing spherical coordinates on $K$, we find

$$\iint_K \frac{1}{r} \frac{\partial u}{\partial n} dS = h \iint \frac{\partial u}{\partial n} \sin\theta \, d\theta \, d\phi \to 0 \text{ as } h \to 0,$$

since $u$ is regular in $V$, and

$$\iint_K - u \frac{\partial \left( \frac{1}{r} \right)}{\partial n} dS = - \iint_K \frac{u}{r^2} dS = - \iint_K u(h, \theta, \phi) \sin\theta \, d\theta \, d\phi$$
$$\to -4\pi u_P$$

where $u_P$ is the value of $u$ at the point $P$. We have therefore

$$(34)\qquad 4\pi u_P = \iint_S \left\{ \frac{1}{r}\frac{\partial u}{\partial n} - u\frac{\partial\left(\frac{1}{r}\right)}{\partial n} \right\} dS - \iiint_V \frac{1}{r}\nabla^2 u\, dV.$$

If $u$ is harmonic in $V$, we obtain the fundamental formula

$$(35)\qquad u_P = \frac{1}{4\pi}\iint_S \left\{ \frac{1}{r}\frac{\partial u}{\partial n} - u\frac{\partial\left(\frac{1}{r}\right)}{\partial n} \right\} dS$$

which was likewise obtained by Green. This is the *third Green's formula.*

If $P$ lies on the surface $S$, at a regular point of the surface, then

$$u_P = \frac{1}{2\pi}\iint_S \left\{ \frac{1}{r}\frac{\partial u}{\partial n} - u\frac{\partial\left(\frac{1}{r}\right)}{\partial n} \right\} dS.$$

This is easily obtained by surrounding $P$ with a small sphere $K$ of radius $h$, calling $K_1$ that portion of the sphere lying in $V$ and $S_1$ that portion of $S$ which lies outside of $K$, applying (26) to the region bounded by $K_1$ and $S_1$ and letting $h \to 0$. We note that in the proof the integral $\iint u(h, \theta, \phi) \sin\theta d\theta d\phi$ is taken only over the half of the unit-sphere.

From equation (35) we can at once see that: the value of a harmonic function in the interior of the region $V$ where it is regular is determined if one knows the value of the function and its normal derivative on the boundary. We remember it was assumed that $u$ and $\frac{\partial u}{\partial n}$ are continuous on approaching

the boundary $S$. Furthermore, equation (35) states that: *every regular harmonic function can be represented as the sum of potentials due to a surface distribution and to a double layer on the surface.*[1] (Compare Chapter II, Art. 7, where this theorem was mentioned.) A word of warning may be needed here. It is not correct to assume that both the value of a potential function and of its normal derivative can be arbitrarily prescribed on $S$, say as continuous functions there. It will later (Chapter VII) be found that merely *either* $u$ *or* $\dfrac{\partial u}{\partial n}$ may be arbitrarily prescribed on $S$. If one places arbitrary continuous functions $f$ and $g$ in (35) for $u$ and $\dfrac{\partial u}{\partial n}$, this integral does indeed define a function $w$ which is regular and harmonic in the interior of $S$. But in general when $P$ approaches $S$, $w$ does not approach the value of $f$ nor does $\dfrac{\partial w}{\partial n}$ approach $g$.

If we put $v = \dfrac{1}{r}$ in (25), we find

$$(36)\quad \iiint_V \left\{ \frac{\partial u}{\partial \xi}\frac{\partial\left(\frac{1}{r}\right)}{\partial \xi} + \frac{\partial u}{\partial \eta}\frac{\partial\left(\frac{1}{r}\right)}{\partial \eta} + \frac{\partial u}{\partial \zeta}\frac{\partial\left(\frac{1}{r}\right)}{\partial \zeta} \right\} dV = \iint_S u\frac{\partial\left(\frac{1}{r}\right)}{\partial n}\,dS + \begin{cases} 4\pi u_P \\ 2\pi u_P, \\ 0 \end{cases}$$

according to the cases where $P$ is in $V$, on $S$, or lies outside of $V$.

By using Green's formula (26*) in the plane with

[1] Here $\dfrac{1}{4\pi}\dfrac{\partial u}{\partial n}$ is the density of the surface mass distribution and $\dfrac{-u}{4\pi}$ is the moment of the double layer.

$$v = \log \frac{1}{r}, \text{ where } r = \sqrt{(x-\xi)^2 + (y-\eta)^2},$$

and $u$ a regular harmonic function of two variables, we obtain

$$(37) \qquad 0 = \oint_C \left\{ u \frac{\partial \left(\log \frac{1}{r}\right)}{\partial n} - \log \left(\frac{1}{r}\right) \cdot \frac{\partial u}{\partial n} \right\} ds$$

when P:$(x, y)$ lies outside the region $S$, and

$$(38) \qquad u_P = \frac{1}{2\pi} \oint_C \left\{ \log \left(\frac{1}{r}\right) \cdot \frac{\partial u}{\partial n} - u \frac{\partial \left(\log \frac{1}{r}\right)}{\partial n} \right\} ds$$

when $P$ lies in the region $S$. If $P$ lies on the boundary $C$, at a regular point, we get

$$(38') \qquad u_P \quad \frac{1}{\pi} \oint_C \left\{ \log \left(\frac{1}{r}\right) \cdot \frac{\partial u}{\partial n} - u \frac{\partial \left(\log \frac{1}{r}\right)}{\partial n} \right\} ds.$$

In proving (38), the point $P$ is excluded from the region of integration by enclosing it in a small circle of radius $h$, and then $h$ is made to approach zero. The factor $\frac{1}{2\pi}$ enters here instead of the $\frac{1}{4\pi}$ of (35), since $2\pi$ is the circumference of the unit circle.

Equations (35) and (38) can be differentiated under the integral sign as many times as desired with respect to the coordinates of $P$ (compare the end of Chapter II, Art. 4), since $P$ does not lie on $S$ (or $C$). *Every potential or harmonic function therefore possesses continuous partial derivatives of arbitrarily high order through the interior of its region of regularity.* This

can be stated in another way, to bring out its significant character: if a function has continuous second partial derivatives in a region and satisfies Laplace's equation there, it possesses continuous derivatives of all orders there.

Every derivative also satisfies Laplace's equation, as is easily seen, and therefore is itself a regular harmonic function.

We will now extend the fundamental equations (35) and (38) with the necessary modifications to infinite regions. For example, suppose that $u$ is a logarithmic potential regular outside the closed curve $C$. Let $K$ be a circle about $P$ (in the region of regularity) of radius $R$ large enough to contain $C$ in its interior, then

$$u_P = \frac{1}{2\pi}\oint_C \left\{ \log\frac{1}{r}\cdot\frac{\partial u}{\partial n} - u\frac{\partial\left(\log\frac{1}{r}\right)}{\partial n}\right\} ds + \frac{1}{2\pi}\oint_K \left\{ \log\frac{1}{r}\cdot\frac{\partial u}{\partial n} - u\frac{\partial\left(\log\frac{1}{r}\right)}{\partial n}\right\} ds.$$

Now

$$\oint_K \log\frac{1}{r}\,\frac{\partial u}{\partial n}\,ds = -R\log R\int_0^{2\pi}\frac{\partial u}{\partial n}\,d\phi.$$

Because $R^2\dfrac{\partial u}{\partial n}$ is bounded (Chapter II, Art. 7), it follows that $R\cdot\log R\cdot\dfrac{\partial u}{\partial n}\to 0$ as $R\to\infty$; hence the limit of the above integral is zero.

Moreover,

$$-\frac{1}{2\pi}\oint_K u\frac{\partial\left(\log\frac{1}{r}\right)}{\partial n}\,ds = \frac{1}{2\pi}\int_0^{2\pi} u\,\frac{\partial(\log r)}{\partial r}\,r\,d\phi = \frac{1}{2\pi}\int_0^{2\pi} u(R,\phi)\,d\phi.$$

Since $u \to c$ as $R \to \infty$, this integral has the limit $c$. Hence we finally find

$$(39) \qquad u_P = \frac{1}{2\pi} \oint_C \left\{ \log \frac{1}{r} \frac{\partial u}{\partial n} - u \frac{\partial \left( \log \frac{1}{r} \right)}{\partial n} \right\} ds + c.$$

In the case of Newtonian potential, equation (35) holds unchanged for infinite regions, that is

$$(40) \qquad u_P = \frac{1}{4\pi} \iint_S \left\{ \frac{1}{r} \frac{\partial u}{\partial n} - u \frac{\partial \left( \frac{1}{r} \right)}{\partial n} \right\} dS,$$

where $u$ is regular in the region outside the closed surface $S$, $P$ is in this exterior region, and the normal is outward from the region and hence inward on $S$. The simple proof of this result is left to the reader.

If we multiply (40) by $\rho = \sqrt{x^2 + y^2 + z^2}$ and let $P:(x, y, z)$ go to infinity, it is easily found that

$$\lim \rho u = \frac{1}{4\pi} \iint_S \frac{\partial u}{\partial n} dS.$$

It is only necessary to note that $\rho/r \to 1$ uniformly for all points $Q$ on $S$, and that

$$\frac{\partial \left( \frac{1}{r} \right)}{\partial n} = - \frac{1}{r^2} \frac{\partial r}{\partial n}$$

vanishes like $1/\rho^2$ with increasing $\rho$. By comparing with Chapter II, Art. 7 (41), we find that

$$\iint_S \frac{\partial u}{\partial n} dS = 4\pi M.$$

The formula (39) remains valid when $u$ is not regular at infinity, but can be represented in the form $u = M \log \dfrac{1}{R} + v$

where $v$ is regular at infinity. Here $u$ is said to have the mass $M$; in this case $c$ is the value of $\lim v$ at infinity. It is sufficient to prove that here also the limit of the integral over $K$ is $c$. Since the mass $M$ is independent of the choice of co-ordinate system, we can write $u$ under the integral sign in the form

$$u = M \log \frac{1}{r} + v_1,$$

thus making $P$ the origin. Then $v_1 \to c$, and

$$\lim_{R\to\infty} \frac{1}{2\pi} \oint_K \left\{ \log \frac{1}{r} \frac{\partial u}{\partial n} - u \frac{\partial \left( \log \frac{1}{r} \right)}{\partial n} \right\} ds$$

$$= \lim_{R\to\infty} \frac{1}{2\pi} \oint \left\{ M \log \frac{1}{r} \frac{\partial \left( \log \frac{1}{r} \right)}{\partial n} - M \log \frac{1}{r} \frac{\partial \left( \log \frac{1}{r} \right)}{\partial n} \right\} ds$$

$$+ \lim_{R\to\infty} \frac{1}{2\pi} \oint \left\{ \log \frac{1}{r} \frac{\partial v_1}{\partial n} - v_1 \frac{\partial \left( \log \frac{1}{r} \right)}{\partial n} \right\} ds$$

$$= 0 + \lim_{R\to\infty} v_1 = c.$$

## Art. 8. Gauss's Mean Value Theorem

Let $P$ and a sphere $K$ about $P$ as centre lie entirely in the region where the function $u$ is a regular harmonic function. Let us apply the formula (35) to this region. This gives

$$u_P = \frac{1}{4\pi} \iint_K \left\{ \frac{1}{r} \frac{\partial u}{\partial n} - u \frac{\partial \left( \frac{1}{r} \right)}{\partial n} \right\} dS;$$

but (letting $h$ be the radius of the sphere)

$$\iint\limits_K \frac{1}{r}\frac{\partial u}{\partial n}dS = \frac{1}{h}\iint\limits_K \frac{\partial u}{\partial n}dS = 0$$

from (29), and

$$-\iint\limits_K u\frac{\partial\left(\frac{1}{r}\right)}{\partial n}dS = \iint\limits_K \frac{u}{h^2}dS$$

so that

$$u_P = \frac{1}{4\pi h^2}\iint\limits_K u dS. \tag{41}$$

Since $\displaystyle\iint\limits_K dS = 4\pi h^2$, we can state the *mean value theorem*: *The value of the Newtonian potential at the centre of a sphere in the region of regularity of the potential is the average of its values over the surface of the sphere.*

The corresponding theorem for logarithmic potential is obtained by taking a circle $K$ of radius $h$ about $P$, small enough so that $u$ is regular and harmonic in and on the circle. Then

$$u_P = \frac{1}{2\pi h}\oint\limits_K u ds. \tag{42}$$

From the mean value theorem it follows that, except for the trivial case of a constant, a potential function cannot have a maximum or minimum value at any point in the interior of the region where it is harmonic, and hence that the extremes of a potential function must lie on the boundary of the region of regularity. For example, suppose that the logarithmic potential $u$ were regular and had a maximum at $P$; then on a sufficiently small circle $C$ about $P$ we would have $u_Q < u_P$ for all points $Q$ on the circle, and the application of (42) gives a contradiction. Also, suppose we assume an improper maxi-

mum, so that $u_Q \leqq u_P$ holds for some region about $P$; then the application of (42) to small spheres about $P$ shows that we must have $u_Q = u_P$ for all points $Q$ in some neighbourhood of $P$. Then from the analytic character of the potential function (to be proved in the next chapter), it follows that $u$ must be a constant through the region of regularity. The proof for Newtonian potential is exactly similar.

Hence, except for the trivial case of a constant function, *a potential regular in the region $V$ can have neither maximum nor minimum in the interior of $V$.*

# CHAPTER IV

## ANALYTIC CHARACTER OF THE POTENTIAL SPHERICAL HARMONICS

### Art. 1. Analytic Character of the Potential

We will now prove that: *every harmonic function can be expanded in a power series in the neighbourhood of any point in free space* (that is, any point outside the masses producing the potential, or any point in the region of regularity of the function); in other words, *a potential function is a regular analytic function at every point in free space.*

We will begin with the expansion of

$$(1) \qquad \frac{1}{r} = \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}},$$

the potential of a unit mass concentrated in the point $Q:(\xi, \eta, \zeta)$. We will expand this in the neighbourhood of an arbitrary point $O$, which can be taken as the origin.[1] Let

$$\overline{OQ} = \sqrt{\xi^2 + \eta^2 + \zeta^2} = l \text{ and } \overline{OP} = \sqrt{x^2 + y^2 + z^2} = h;$$

we seek a development valid in a small sphere about the origin, say the sphere with radius $h \leq \frac{l}{8}$. We write (1) in the form

$$(1^*) \qquad \frac{1}{r} = \frac{1}{\sqrt{l^2 - [(2x\xi + 2y\eta + 2z\zeta) - h^2]}} = \frac{1}{l} \quad \frac{1}{(1-q)^{\frac{1}{2}}}$$

where

$$(2) \qquad q = \frac{2x\xi + 2y\eta + 2z\zeta}{l^2} - \frac{h^2}{l^2}.$$

[1]Of course, the point $O$ must not coincide with $Q$.

The development in the binomial series

$$(3) \qquad (1-q)^{-\frac{1}{2}} = 1 + \tfrac{1}{2}q + \frac{1\cdot 3}{2\cdot 4}q^2 + \ldots = \sum_{n=0}^{\infty} c_n q^n$$

converges for $|q| < 1$. Here the general coefficient is

$$(4) \qquad c_n = \binom{-\frac{1}{2}}{n}(-1)^n = \frac{1\cdot 3\ldots(2n-1)}{2\cdot 4\ldots 2n} = \frac{(2n)!}{2^{2n}(n!)^2}.$$

Setting

$$p = 2\,\frac{|x|+|y|+|z|}{l} + \frac{h^2}{l^2},$$

and noting that $\dfrac{|\xi|}{l} \leq 1, \ldots, \dfrac{|\zeta|}{l} \leq 1$, we have

$$|q| \leq p < \frac{6h}{l} + \frac{h^2}{l^2} < 7\,\frac{h}{l} \leq \frac{7}{8}.$$

Hence finally we have the expansion

$$(5) \qquad \frac{1}{r} = \frac{1}{l}\left(1 + \frac{q}{2} + \frac{1\cdot 3}{2\cdot 4}q^2 + \ldots\right)$$

$$= \frac{1}{l}\left(1 + \frac{x\xi}{l^2} + \frac{y\eta}{l^2} + \frac{z\zeta}{l^2} + \ldots\right)$$

where the dots represent terms of the second and higher orders in $x, y, z$.

It remains to be proved that the power series in $x, y, z$ on the right in (5) is absolutely convergent for $h \leq \dfrac{l}{8}$, so that the rearrangement is justified.

First of all we have

$$\frac{1}{l}(1-p)^{-\frac{1}{2}} = \frac{1}{l}\left(1 + \tfrac{1}{2}p + \frac{1\cdot 3}{2\cdot 4}p^2 + \ldots\right),$$

and this series of positive terms is convergent for $h \leq \dfrac{l}{8}$, since $p < \frac{7}{8} < 1$. Moreover, we can write

$$\frac{1}{l}(1-p)^{-\frac{1}{2}} = \frac{1}{l}\left[1 + \tfrac{1}{2}\left(2\frac{|x|+|y|+|z|}{l} + \frac{h^2}{l^2}\right) + \right.$$

$$\left. \frac{1\cdot 3}{2\cdot 4}(\ldots)^2 + \ldots\right] = \sum b_{mnr}|x|^m|y|^n|z|^r \tag{6}$$

$$(m, n, r = 0, 1, 2, \ldots)$$

where the $b_{mnr}$ are obviously positive. This rearrangement can be made, since all terms in the brackets are positive. And the power series in $|x|$, $|y|$, $|z|$ is of course convergent for $h \leq \dfrac{l}{8}$.

If we write (5) in the form

$$\frac{1}{r} = \sum a_{mnr}\, x^m y^n z^r \tag{5*}$$

it is evident that this series is dominated by the series on the right in (6) since the two series are derived in exactly the same way from $(1-q)^{-\frac{1}{2}}$ and $(1-p)^{-\frac{1}{2}}$ respectively, and since $p$ is derived from $q$ by replacing the terms of $q$ by their absolute values and then replacing $\dfrac{|\xi|}{l}$, $\dfrac{|\eta|}{l}$, $\dfrac{|\zeta|}{l}$ by 1, which is not smaller. Therefore indeed $|a_{mnr}| \leqq b_{mnr}$ for all $m$, $n$, and $r$. Accordingly (5), or (5*), is certainly valid for $h \leq \dfrac{l}{8}$. The convergence is uniform in the sphere $h \leq \dfrac{l}{8}$; for the dominant series in (6) is uniformly convergent. This last statement follows from the fact that the series in (6) is dominated by the convergent series of constant terms

$$\frac{1}{l}\left[c_0 + c_1\frac{7}{8} + c_2\left(\frac{7}{8}\right)^2 + \ldots\right].$$

The expansion (5*) is, of course, identical with the Taylor expansion; thus

$$a_{mnr} = \frac{1}{m!n!r!} \left. \frac{\partial^{m+n+r}\left(\frac{1}{r}\right)}{\partial x^m \partial y^n \partial z^r} \right|_{(x,\, y,\, z) = (0,\, 0,\, 0.)}$$

The coefficients are functions of $\xi$, $\eta$, $\zeta$. Such an expansion is possible in only one way, as is shown in the theory of power series.

The expansion of the potential due to a distribution,

$$u = \iiint\limits_V \frac{\rho}{r} dV$$

where $\rho$ is bounded and integrable, is now very simple. If $O$ is an arbitrary point of free space (hence outside $V$), which we take as origin, and if $l_1$ is the smallest distance from $O$ to any point $Q$ of $V$, then the expansion (5) is certainly valid for $P$ anywhere inside the sphere $S$ of centre $O$ and radius $\frac{l_1}{8}$, and the series converges uniformly for all points $P$ in $S$ and all points $Q$ in $V$, because it has a dominating series (6). On account of this uniform convergence, we can multiply the series (5) by $\rho$ and integrate termwise over the region $V$. We thus obtain a power series expansion for $u$, which converges uniformly in $S$.

In an exactly similar manner we can prove that the Newtonian potential due to a surface distribution, $\iint\limits_S \frac{\sigma}{r} dS$, can be expressed as a power series.

For the potential of a double layer, we need the development of

$$\frac{\partial\left(\frac{1}{r}\right)}{\partial n}$$

where the point $Q$ is variable when calculating the derivative in the direction $n$. Now

$$\frac{\partial\left(\dfrac{1}{r}\right)}{\partial n} = \frac{\partial\left(\dfrac{1}{r}\right)}{\partial \xi}\frac{\partial \xi}{\partial n} + \ldots + \frac{\partial\left(\dfrac{1}{r}\right)}{\partial \zeta}\frac{\partial \zeta}{\partial n}$$

$$= -\left[\frac{\partial\left(\dfrac{1}{r}\right)}{\partial x}\frac{\partial \xi}{\partial n} + \cdots\right],$$

since

$$\frac{\partial r}{\partial \xi} = -\frac{\partial r}{\partial x}, \ldots .$$

But the three partial derivatives of $\dfrac{1}{r}$ with respect to $x$, $y$, $z$ can be expanded in power series. For example, by differentiating (5), we get

$$\frac{\partial\left(\dfrac{1}{r}\right)}{\partial x} = \frac{1}{l}\left(c_1 + 2c_2 q + 3c_2 q^2 + \ldots\right)\frac{\partial q}{\partial x}$$

and the series in the parentheses is absolutely and uniformly convergent for all points $P$ in $S$ and $Q$ in $V$, since it is dominated by the convergent series (independent of $P$ and $Q$)

$$c_1 + 2c_2\left(\frac{7}{8}\right) + 3c_3\left(\frac{7}{8}\right)^2 + \ldots .$$

And after multiplying this series by

$$\frac{1}{l}\frac{\partial q}{\partial x} = \frac{2(\xi - x)}{l^2}\cdot\frac{1}{l},$$

the result is still uniformly and absolutely convergent. The rearrangement into a power series in $x$, $y$, $z$ can be justified as above for $\dfrac{1}{r}$ itself.

Naturally it is possible to prove in a similar manner that the higher order derivatives of $\dfrac{1}{r}$ can be obtained, by termwise differentiation, as absolutely and uniformly convergent series.

Thus we find that

$$\frac{\partial\left(\frac{1}{r}\right)}{\partial n}$$

can be expanded in a power series, absolutely and uniformly convergent for all points $P$ of a sphere about the origin and all points $Q$ of the surface carrying the double layer; then by termwise integration, we obtain the power series expansion for the potential

$$u = \iint_S \mu \frac{\partial\left(\frac{1}{r}\right)}{\partial n} dS$$

of a double layer.

It is thus proved that any Newtonian potential can be represented by a power series; but since we proved that any harmonic function is a potential function (Chapter III, Art. 7), it follows that any regular solution of Laplace's equation is analytic (representable by power series) in its domain of regularity.

The series expansion of a harmonic function $u$ has the form

(7) $$u = \sum c_{mnr} x^m y^n z^r, \quad (m, n, r = 0, 1, 2, 3, \ldots)$$

$$c_{mnr} = \frac{1}{m!\,n!\,r!} \left.\frac{\partial^{m+n+r} u}{\partial x^m \partial y^n \partial z^r}\right|_{(x, y, z) = (0, 0, 0)},$$

and the coefficients are uniquely determined.

The series can be differentiated termwise arbitrarily often. If the development is to be about the point $(x_0, y_0, z_0)$ in the domain of regularity of the function, the expansion is a Taylor series in $x - x_0$, $y - y_0$, $z - z_0$.

For the logarithmic potential, a power series expansion of the form

(7*) $$u = \sum c_{mn} x^m y^n,$$

about an arbitrary point taken as the origin, is obtained in a similar manner.

From this series development it follows that: *if a function is harmonic and regular in $V$ and vanishes at all points in a subregion $V_1$, then it vanishes identically over the whole region $V$.* The proof is by the process of analytic continuation, often used in function theory. Thus, let $O$ be a point in $V_1$ and $P$ an arbitrary point of $V$. Construct a chain of circles from $O$ to $P$, such that each circle contains the centre of the next one.[2] Now, by hypothesis, the series expansion vanishes identically in the first circle, and therefore in the second, third, etc. Hence the potential must vanish at $P$.

If two potentials are identical in value over a subregion of their common region of regularity, they are identical over the whole of this common region, since their difference must vanish there. If a potential is constant over a portion of its region of regularity, it is constant over the whole region.

## Art. 2. Expansion of $\frac{1}{r}$ in Spherical Harmonics. Legendre Polynomials

We will now carry through the expansion of $1/r$ in a modified form. Let (Fig. 6)

$$\overline{OP} = \sqrt{x^2 + y^2 + z^2} = \rho, \quad \overline{OQ} = \sqrt{\xi^2 + \eta^2 + \zeta^2} = l,$$
$$\overline{PQ} = \sqrt{(\xi - x)^2 + \dots} = r$$

and let $u = \cos\alpha$ where $\alpha$ is the angle between $\overline{OP}$ and $\overline{OQ}$. Hence $|u| \leq 1$. We will expand in powers of $\rho$. We have

$$(8) \qquad u = \cos\alpha = \frac{x\xi + y\eta + z\zeta}{\rho l}$$
$$= \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi - \varphi'),$$

[2]The first circle lies entirely in $V_1$ and its centre is $O$, the last circle contains $P$. For space, one has to use spheres instead of circles.

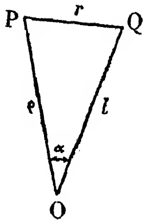

FIG. 6

where $\rho$, $\theta$, $\varphi$ and $l$, $\theta'$, $\varphi'$ are the spherical coordinates of $P$ and $Q$, and also

(9) $$r^2 = l^2 - 2\rho l u + \rho^2,$$

so that

(10) $$\frac{1}{r} = \frac{1}{l}\left(1 - \frac{2\rho u}{l} + \frac{\rho^2}{l^2}\right)^{-\frac{1}{2}}.$$

Let $$q = \frac{2\rho u}{l} - \frac{\rho^2}{l^2};$$

then from (3) we have, for $|q| < 1$,

(11) $$\left(1 - \frac{2\rho u}{l} + \frac{\rho^2}{l^2}\right)^{-\frac{1}{2}} = \sum c_m\left(\frac{2\rho u}{l} - \frac{\rho^2}{l^2}\right)^m.$$

This series is dominated by the series

$$\sum c_m\left(\frac{2\rho}{l} + \frac{\rho^2}{l^2}\right)^m$$

(since $|u| \leq 1$), and this dominating series converges for $\rho$ sufficiently small; for example, for $\rho < \frac{l}{4}$, since

$$\left(\frac{2\rho}{l} + \frac{\rho^2}{l^2}\right) < \frac{3\rho}{l} < \frac{3}{4} < 1.$$

One may therefore rearrange the dominating series in powers of $\rho$, because the terms in parentheses are positive, and finally (compare Art. 1) the terms in (11) may likewise be arranged

in powers of $\rho$. This gives the series, certainly absolutely convergent if $\rho < \frac{l}{4}$,

$$(12)\qquad \left(1 - \frac{2\rho u}{l} + \frac{\rho^2}{l^2}\right)^{-\frac{1}{2}} = P_0(u) + P_1(u)\cdot\left(\frac{\rho}{l}\right) + P_2(u)\cdot\left(\frac{\rho}{l}\right)^2 + \ldots,$$

if we denote the coefficients, which depend on $u$, by $P_n(u)$. In order to calculate $P_n(u)$, we expand the general term from (11)

$$c_m\left(\frac{2\rho u}{l} - \frac{\rho^2}{l^2}\right)^m = c_m \sum_{k=0}^{m} (-1)^k \binom{m}{k}\left(\frac{2\rho u}{l}\right)^{m-k}\left(\frac{\rho}{l}\right)^{2k};$$

hence

$$(11^*)\qquad \left(1 - \frac{2\rho u}{l} + \frac{\rho^2}{l^2}\right)^{-\frac{1}{2}} = \sum_{m=0}^{\infty}\sum_{k=0}^{m} (-1)^k \frac{(2m)!}{2^{2m}(m!)^2}\binom{m}{k} 2^{m-k}u^{m-k}\left(\frac{\rho}{l}\right)^{m+k}$$

From these terms, we must select the terms in $\frac{\rho}{l}$ which have the same exponent $n$, i.e. those for which $m+k=n$ or $m=n-k$, and add them together. This gives

$$P_n(u) = \sum_k (-1)^k \frac{(2n-2k)!}{2^{2n-2k}((n-k)!)^2}\binom{n-k}{k} 2^{n-2k}\, u^{n-2k}$$

where $k$ runs through the integers $0, 1, 2, \ldots$ up to $[\frac{1}{2}n]$.[3] The coefficient $b_{n-2k}$ of $u^{n-2k}$ in $P_n(u)$ is finally

$$b_{n-2k} = \frac{1}{2^n}\frac{(-1)^k(2n-2k)!}{(n-k)!k!(n-2k)!}\,;\text{in particular, } b_n = \frac{(2n)!}{2^n(n!)^2}\cdot$$

Thus

[3] $[\frac{1}{2}n]$ means the largest integer contained in $\frac{1}{2}n$, e.g. $[\frac{1}{2}9]=4$.

$$(13) \qquad P_n(u) = \frac{1}{2^n} \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k (2n-2k)!}{k!\,(n-k)!\,(n-2k)!} u^{n-2k}.$$

Hence $P_n(u)$ is a polynomial of the $n$th degree in $u$. It is known as a *Legendre Polynomial.* The polynomials with $n$ even contain only even powers, and those with $n$ odd contain only odd powers of $u$. We have in particular

$$(13^*) \qquad P_0(u) = 1,\ P_1(u) = u,\ P_2(u) = \tfrac{3}{2}(u^2 - 1/l), \quad P_3(u) = \tfrac{5}{2}(u^3 - \tfrac{3}{5}u),\ \text{etc.}$$

For later convenience, we will give a different derivation of the Legendre polynomials. If we set $\dfrac{\rho}{l} = v$ for brevity, we have to expand the function (neglecting the factor $1/l$)

$$(1 - 2uv + v^2)^{-\frac{1}{2}}.$$

The polynomial in the parenthesis may be factored; we easily find that

$$(1 - 2uv + v^2) = (1 - e^{-ai}v)(1 - e^{ai}v),$$

since $2u = 2\cos a = e^{ai} + e^{-ai}$. Hence for $|v| < 1$, we have

$$\begin{aligned}(1-2uv+v^2)^{-\frac{1}{2}} &= (1-ve^{ai})^{-\frac{1}{2}}\cdot(1-ve^{-ai})^{-\frac{1}{2}} \\ &= (c_0+c_1ve^{ai}+c_2v^2e^{2ai}+\ldots)(c_0+c_1ve^{-ai}+\ldots) \\ &= c_0^2+c_0c_1(e^{ai}+e^{-ai})v+[c_0c_2(e^{2ai}+e^{-2ai})+c_1^2]v^2+\ldots \\ &= c_0^2+(2c_0c_1\cos a)v+(2c_0c_2\cos 2a+c_1^2)v^2+\ldots\end{aligned}$$

Comparison with the series (12) gives

$$P_0(u) = c_0^2,\ P_1(u) = 2c_0c_1\cos a,\ P_2(u) = 2c_0c_2\cos 2a + c_1^2, \ldots$$

and in general

$$(14) \qquad P_n(u) = P_n(\cos a) = 2c_0c_n\cos na + 2c_1c_{n-1}\cos(n-2)a + 2c_2c_{n-2}\cos(n-4)a + \ldots.$$

For $n$ even, the last term is $(c_{\frac{1}{2}n})^2$, and not $2\,(c_{\frac{1}{2}n})^2$. In this formula the Legendre polynomials with argument $\cos a$ are expressed in terms of the cosines of multiples of $a$. The coeffi-

cients of these sums are all positive; hence the maximum value of $|P_n(u)|$ is obtained when $\alpha = 0$ or $\alpha = \pi$, that is, when $u = 1$ or $u = -1$. But for $u = 1$,

$$(1 - 2uv + v^2)^{-\frac{1}{2}} = \frac{1}{1 - v} = 1 + v + v^2 + \dots.$$

From this it follows that $P_n(1) = 1$. In a similar manner, it is found that $P_n(-1) = (-1)^n$. Accordingly, when $u$ is restricted to the interval $(-1, 1)$, $|P_n(u)|$ has the maximum value of 1 for all $n$. This fact will be used later.

We will prove (in Art. 6 and Exercises of this Art.) that the derivative $\dfrac{dP_n(u)}{du}$, which is evidently a polynomial of degree $n - 1$, can be expressed in terms of the Legendre polynomials up to order $n - 1$, by the equation

$$\frac{dP_n(u)}{du} = (2n - 1)P_{n-1}(u) + (2n - 5)P_{n-3}(u) + (2n - 9)P_{n-5}(u) + \dots. \tag{15}$$

If we accept this formula at present, it follows that $\left|\dfrac{dP_n}{du}\right|$ also reaches its maximum at $u = 1$ and $u = -1$, and that this maximum is $(2n - 1) + (2n - 5) + \dots$, hence certainly $< n^2$. Therefore $\left|\dfrac{dP_n}{\partial u}\right| < n^2$ in the whole interval $(-1, 1)$.

By differentiation of the above identity, we find

$$\frac{d^2P_n}{du^2} = (2n - 1)\frac{dP_{n-1}}{du} + (2n - 5)\frac{dP_{n-3}}{du} + \dots$$

so that $\left|\dfrac{d^2P_n(u)}{du^2}\right| < n^4$ in the interval $-1 \leq u \leq 1$. Similar bounds are obtained in the same manner for the higher derivatives.

From (10) and (12),

$$\frac{1}{r} = \sum_{n=0}^{\infty} P_n(u) \frac{\rho^n}{l^{n+1}} \tag{16}$$

is certainly valid for $\frac{\rho}{l} < \frac{1}{4}$. The general term of this series

$$P_n(u) \frac{\rho^n}{l^{n+1}} = F_n(x, y, z) \tag{17}$$

is a homogeneous polynomial of the $n$th degree in $x, y, z$. For $P_n(u)$ contains only the powers $u^n, u^{n-2}, u^{n-4}, \ldots$ and hence from (8) contains $\rho$ only in the powers $\rho^{-n}, \rho^{-n+2}, \ldots$. Hence $P_n(u)\rho^n$ contains $\rho$ only in the even powers $\rho^0, \rho^2, \rho^4, \ldots$. Hence $P_n\rho^n$ is rational and integral of degree $n$ in $x, y, z$. Moreover $u$ and hence also $P_n(u)$ is homogeneous of degree zero. Accordingly $P_n(u)\rho^n$ and hence also $F_n(x, y, z)$ are actually homogeneous of degree $n$. The coefficients of these polynomials depend on $\xi, \eta, \zeta$.

If we arrange

$$\frac{1}{r} = F_0 + F_1 + \ldots + F_n + \ldots, \tag{18}$$

according to the powers and products of $x, y, z$, we obtain again the series development (5*), which is necessarily the Taylor series; we have proved before that this converges in a sufficiently small neighbourhood of $O$, say for $\frac{\rho}{l} < \frac{1}{8}$.

The series (18) converges in a larger region than (5*). Since $|P_n(u)| \leq 1$ for $|u| \leq 1$, it follows immediately that: *the series* (18) *converges absolutely for* $\frac{\rho}{l} < 1$, *that is, for all points* $P$ *in the interior of a sphere about* $O$ *of radius* $\overline{OQ} = l$, *and it converges uniformly for any closed region lying inside this sphere.*

When $\frac{\rho}{l} < \frac{1}{8}$, the series (5*) for $\frac{1}{r}$ may be differentiated termwise arbitrarily often. This is permissible also for (18), at least for $\frac{\rho}{l} < \frac{1}{8}$, since the combining of powers and products is naturally permitted. Hence, if one forms the differential expression of Laplace, it follows that

$$\nabla^2 \frac{1}{r} = 0 = \nabla^2 F_0 + \nabla^2 F_1 + \dots + \nabla^2 F_n + \dots.$$

For $n \geqq 2$, $\nabla^2 F_n$ of this series is a homogeneous polynomial of degree $n-2$. By separating each term of this series into its parts, a power series in $x, y, z$ is obtained. Since this series converges in a certain sphere about the origin and has the value zero there, all its coefficients must be zero. Hence all the terms $\nabla^2 F_n$ must vanish, so that

$$\nabla^2 F_n(x, y, z) = 0. \tag{19}$$

All terms of (18) are therefore solutions of Laplace's equation. A homogeneous polynomial which satisfies Laplace's equation is called a *spherical harmonic*. The degree of the polynomial is the *order* of the spherical harmonic. The functions $F_n(x, y, z)$ defined in (17) are therefore spherical harmonics of order $n$, and (18) is the expansion of $\frac{1}{r}$, that is of the potential of a unit mass, in spherical harmonics.

It is now not difficult to obtain the expansion of the derivatives of $\frac{1}{r}$ in spherical harmonics. The termwise differentiation, which is certainly permissible in a sufficiently small sphere about the origin, gives for the derivative with respect to $x$

$$(16^*) \qquad \frac{\partial\left(\frac{1}{r}\right)}{\partial x} = \sum_n \frac{1}{l^{n+1}} \frac{\partial}{\partial x} (P_n(u)\rho^n).$$

But

$$\frac{\partial}{\partial x}(P_n(u)\rho^n) = \rho^n P_n'(u)\frac{\xi}{\rho l} - \rho^n P_n'(u)\frac{ux}{\rho^2} + n\rho^{n-2} x P_n(u)$$

and since

$$\frac{|\xi|}{l} \leq 1, \frac{|x|}{\rho} \leq 1, |u| \leq 1, P_n(u) \leq 1, |P_n'(u)| \leq n^2,$$

$$\left|\frac{\partial}{\partial x}(P_n(u)\rho^n)\right| \leq n^2\rho^{n-1} + n^2\rho^{n-1} + n\rho^{n-1} \leq 3n^2\rho^{n-1}.$$

The series of absolute values of the terms of (16*) therefore has the dominating series

$$\frac{3}{l^2}\sum n^2 \frac{\rho^{n-1}}{l^{n-1}}$$

(of constant terms) which converges for $\frac{\rho}{l} < 1$. Hence the series (16*) converges absolutely and uniformly in the same region as (16) does. The expansion for

$$\frac{\partial\left(\frac{1}{r}\right)}{\partial x},$$

and of course also for

$$\frac{\partial\left(\frac{1}{r}\right)}{\partial y} \text{ and } \frac{\partial\left(\frac{1}{r}\right)}{\partial z},$$

is therefore valid in the same region as (18). The general term $\frac{\partial}{\partial x}(P_n(u)\rho^n) = \frac{\partial F_n}{\partial x}$ is a homogeneous polynomial of the

$(n-1)$st degree, and hence is a spherical harmonic of order $n-1$, since it obviously satisfies the Laplace equation.

It is not difficult to show that also the higher derivatives of $\frac{1}{r}$ may be expanded in the same region in spherical harmonics obtained by termwise differentiation. For instance, the second derivatives of the expansion (18) are series having the dominating series $\sum n^4 \frac{\rho^{n-1}}{l^{n-1}}$ (aside from a constant factor).

EXERCISES

1. *The recursion formulas for the Legendre polynomials.*

By setting $\frac{\rho}{l} = v$ in equation (12) (Art. 2) we get

(12*) $f(u,v) = (1-2vu+v^2)^{-\frac{1}{2}} = (P_0(u) + vP_1(u) + v^2P_2(u) + \ldots.$

Differentiate with respect to $v$, assuming $|v| \leq a < 1$.

$$\frac{\partial f}{\partial v} = \frac{u-v}{(1-2vu+v^2)^{3/2}} = P_1(u) + 2vP_2(u) + 3v^2P_3(u) + \ldots.$$

Termwise differentiation is permissible since the series on the right converges uniformly (also absolutely) in $v$ for $|v| \leq a$ (remember $|P_n(u)| \leq 1$). Accordingly $(1-2vu+v^2)\frac{\partial f}{\partial v} = (u-v)f$

or
$$(1-2vu+v^2)(P_1(u) + 2vP_2(u) + \ldots) =$$
$$(u-v)(P_0(u) + vP_1(u) + v^2P_2(u) + \ldots).$$

Write each side of this equation as a power series in $v$, compare the coefficients of equal powers of $v$ and find

(A) $(n+1)P_{n+1}(u) - (2n+1)uP_n(u) + nP_{n-1}(u) = 0.$

Differentiate (12*) with respect to $u$

$$\frac{\partial f}{\partial u} = \frac{v}{(1-2vu+v^2)^{3/2}} = P_0'(u) + vP_1'(u) + v^2P_2'(u) + \ldots.$$

The series on the right converges uniformly (also absolutely) in $u$ for $|u| \leq 1$, if $|v|$ is assumed to be $<1$ (recalling that

$$|P'_n(u)| \leq n^2).$$

Therefore termwise differentiation is justified. Accordingly

$$(u - v)\frac{\partial f}{\partial u} = v\frac{\partial f}{\partial v}$$

or

$$(u - v)(P'_0(u) + vP'_1(u) + v^2P'_2(u) + \ldots) = v(P_1(u) + 2vP_2(u) + 3v^2P_3(u) + \ldots).$$

From this find

(B) $$uP'_n(u) - P'_{n-1}(u) = nP_n(u).$$

Differentiate (A), replace $P'_{n-1}(u)$ by its value in (B) and find

(C) $$P'_{n+1}(u) - uP'_n(u) = (n + 1)P_n(u).$$

Infer from (B) and (C)

(D) $$P'_{n+1}(u) - P'_{n-1}(u) = (2n + 1)P_n(u).$$

Write $n - 1$ for $n$ in (C)

(C*) $$P'_n - uP'_{n-1} = nP_{n-1}$$

eliminate $P'_{n-1}(u)$ between (C*) and (B), and find

(E) $$(1 - u^2)P'_n(u) = nP_{n-1}(u) - nuP_n(u).$$

2. Infer from (E) and (B) the differential equation (27) of Article 5 by differentiating (E) and eliminating $P'_{n-1}(u)$.

3. Eliminating $P'_n(u)$ between (C) and (B) we obtain $P'_{n+1} = (2n + 1)P_n + P'_{n-1}$ or $P'_n = (2n - 1)P_{n-1} + P'_{n-2}$. From this infer equation (15).

4. Infer from (13) that $P_n(0) = (-1)^n \dfrac{1\cdot3\ldots(2n-1)}{2\cdot4\ldots(2n)}$ when $n$ is even, and $P_{2n+1}(0) = 0$.

## Art. 3. Expansion of the Newtonian Potential

From the expansion of (16) and (16*) it is not difficult to obtain the expansion of all Newtonian potentials in spherical

harmonics. Consider first the potential of a body $U = \iiint\limits_V \frac{\tau}{r} dV$ where the density $\tau$ is again assumed to be bounded and integrable. Let $O$ be an arbitrary point of free space, taken as the origin, and let $h$ be the smallest distance from $O$ to any point $Q$ of $V$. If we take a sphere $F$ about $O$ of radius $k < h$, then the series (16) or (18) is absolutely and uniformly convergent for all points $P$ of $F$ (i.e., $\rho \leq k$) and $Q$ of $V$ (i.e., $l \geq h$), since it has the dominating series of constants (remembering that $|P_n| \leq 1$), $\frac{1}{h}\Sigma\left(\frac{k}{h}\right)^n$. Hence (18) may be integrated termwise after multiplication by $\tau$, and we get

$$(20) \qquad U = \iiint\limits_V \frac{\tau}{r} dV = G_0 + G_1 + \ldots + G_n + \ldots,$$

$$G_n(x, y, z) = \iiint\limits_V \tau F_n dV = \rho^n \iiint\limits_V \frac{\tau P_n(u)}{l^{n+1}} dV,$$

where $dV = d\xi d\eta d\zeta$.

The series (20) is absolutely and uniformly convergent for all points $P$ of $F$, i.e., for all points of any sphere about $O$ which consists entirely of points in free space.

Evidently $G_n$ is a homogeneous polynomial of degree $n$ in $x, y, z$, since this was true for $F_n$, and integration with respect to the integration variables $\xi, \eta, \zeta$ does not affect this property. Moreover it is evident that (20) may be differentiated under the integral sign arbitrarily often with respect to $x, y, z$, the integrand being everywhere bounded and arbitrarily often differentiable. It follows that $G_n$ satisfies the Laplace equation since $F_n$ does. Hence $G_n$ is a spherical harmonic of order $n$.

In a similar manner the potential of a surface distribution can be expanded in spherical harmonics:

$$(21) \qquad \iint\limits_S \frac{\sigma}{r}\, dS = H_0 + H_1 + \ldots + H_n + \ldots,$$

$$H_n(x, y, z) = \rho^n \iint\limits_S \frac{\sigma P_n(u)}{l^{n+1}}\, dS.$$

The series is absolutely and uniformly convergent in any sphere about $O$ which contains only points of free space.

From equation (16*), and the analogous developments of

$$\frac{\partial\left(\frac{1}{r}\right)}{\partial y} \text{ and } \frac{\partial\left(\frac{1}{r}\right)}{\partial z},$$

it is evident that the potential of a double layer can also be expanded in a series of spherical harmonics, which converges in the usual manner,

$$(22) \qquad \iint\limits_S \mu \frac{\partial\left(\frac{1}{r}\right)}{\partial n}\, dS = L_0 + L_1 + \ldots + L_m + \ldots,$$

$$L_m(x, y, z) = \rho^m \iint\limits_S \frac{\mu}{l^{m+1}} \frac{\partial P_m(u)}{\partial n}\, dS.$$

In the differentiation $\frac{\partial}{\partial n}$, $Q: (\xi, \eta, \zeta)$ is, of course, the variable point. The above expansions (20), (21), and (22) can be differentiated arbitrarily often in their regions of convergence. The derived series are convergent in the same regions and represent the derivatives of the corresponding potentials.

Having completed the subject of expansion of potentials, we can consider the expansion of any harmonic function in series of spherical harmonics. However, it is perhaps better to postpone this development and base it on the Poisson integral (Chapter IX) as this leads to an elegant and simple representation.

## Art. 4. Derivation of the Spherical Harmonics in Rectangular Coordinates from Laplace's Equation

As an illustrative exercise we will obtain the spherical harmonics, to which we were led in the series expansion of $\frac{1}{r}$ and the potentials, in cartesian coordinates from Laplace's equation.

Since this is a linear homogeneous differential equation, a linear combination $k_1u_1+\ldots+k_nu_n$ of solutions $u_i$ is a solution. This solution is linearly dependent on $u_1, u_2, \ldots, u_n$. It is, of course, sufficient to find the linearly independent spherical harmonics. Since spherical harmonics of different orders are obviously independent, it is only necessary to find the linearly independent spherical harmonics of arbitrary order $n$.

We have first as spherical harmonics of order 0 the function 1, and for order 1 the functions $x$, $y$, $z$. For those of order 2, we form the homogeneous polynomial

$$F_2(x, y, z) = c_1x^2 + c_2y^2 + c_3z^2 + c_4yz + c_5zx + c_6xy,$$

whose six terms are linearly independent. Now

$$\nabla^2 F_2 = 2(c_1 + c_2 + c_3).$$

For $\nabla^2 F_2 = 0$, it is therefore necessary and sufficient that $c_1 + c_2 + c_3 = 0$. If we eliminate $c_3$, we obtain

$$F_2 = c_1(x^2 - z^2) + c_2(y^2 - z^2) + c_4yz + c_5zx + c_6xy;$$

and this polynomial with 5 arbitrary parameters is a spherical harmonic for any arbitrary choice of the parameters. Hence we have exactly 5 linearly independent spherical harmonics of order 2:

$$x^2 - z^2,\ y^2 - z^2,\ xy,\ yz,\ zx.$$

We proceed in an exactly similar manner for any order $n$. In

Chapter IX, Art. 3, we will prove that there are exactly $2n + 1$ linearly independent spherical harmonics of order $n$. We now give a method of determining such functions. The general homogeneous polynomial $F_n$ of the $n$th degree in three independent variables has $\frac{(n+1)(n+2)}{2} = m$ linearly independent terms (number of combinations with repetitions of $n$ elements of three kinds). Let these terms have the coefficients $c_1, c_2, \ldots, c_m$. Now the Laplacian $\nabla^2 F_n$ of this polynomial $F_n$ is of degree $n - 2$, and therefore has $\frac{(n-1)n}{2} = s$ terms, linearly independent. The coefficients $b_1, b_2, \ldots, b_s$ are linear homogeneous combinations of the constants $c_1, c_2, \ldots, c_m$. For $\nabla^2 F_n = 0$, it is necessary and sufficient that all the coefficients $b_1, b_2, \ldots, b_s$ vanish. We get therefore $s$ linear homogeneous equations for the $m$ coefficients for $c_i$. Then $s$ of the numbers $c_i$ can be expressed as linear homogeneous combinations of the remaining $m - s$, which therefore remain arbitrary. If we eliminate in this manner $s$ of the coefficients $c_i$ of $F_n$, then $F_n$ becomes a spherical harmonic of order $n$, and contains $m - s = 2n + 1$ arbitrary parameters. The polynomials, which are multiplied by these parameters, are the desired linearly independent spherical harmonics.

EXAMPLE: order $n = 3$. Here $m = 10$, $s = 3$, hence $m - s = 7$.

$$F_3 = c_1x^3 + c_2x^2y + c_3x^2z + c_4xy^2 + c_5\,xyz + c_6xz^2 + c_7y^3 + c_8y^2z + c_9yz^2 + c_{10}z^3.$$

Show that

$$F_3 = c_1(x^3 - 3xz^2) + c_2(x^2y - yz^2) + c_3(x^2z - y^2z) + c_4(xy^2 - xz^2) + c_5xyz + c_7(y^3 - 3yz^2) + c_{10}(z^3 - 3y^2z)$$

is a spherical harmonic with 7 arbitrary parameters, and hence contains 7 linearly independent spherical harmonics of order 3.

## Art. 5. Surface Spherical Harmonics. Their Differential Equation. Associated Functions

Laplace's equation in spherical coordinates is[4]

$$\nabla^2 u = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} + \frac{1}{\rho^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) = 0. \tag{23}$$

For these coordinates,

$$x = \rho \sin\theta \cos\phi,\ y = \rho \sin\theta \sin\phi,\ z = \rho\cos\theta.$$

If $F_n\ (x, y, z)$ is any spherical harmonic of order $n$, it can clearly be written in the form

$$F_n(x, y, z) = \rho^n A_n(\theta, \phi), \tag{24}$$

where $A_n$ depends only on $\phi$ and $\theta$, and indeed is a homogeneous polynomial of the $n$-th degree in the trigonometric functions $\sin\theta\cos\phi$, $\sin\theta\sin\phi$, $\cos\theta$. The functions $A_n$ are called *surface spherical harmonics* of order $n$. The equation (24) may be considered the definition of surface spherical harmonics. The entire set of surface spherical harmonics of order $n$ is obtained by expressing the spherical harmonics of this order in spherical coordinates and dividing by $\rho^n$. We find:

order 0: 1,
order 1: $\sin\theta\cos\phi$, $\sin\theta\sin\phi$, $\cos\theta$,
order 2: $\sin^2\theta \sin 2\phi$, $\sin^2\theta\cos 2\phi$, $\sin 2\theta\cos\phi$, $\sin 2\theta \sin\phi$, $1-3\cos^2\theta$, etc.

The functions $A_n$ satisfy a linear homogeneous differential equation of the second order, which is obtained by substituting (24) in (23) and then dividing through by the common factor $\rho^{n-2}$ which occurs in every term. This gives

[4]A derivation of this equation is to be found in W. F. Osgood's *Advanced Calculus*, New York, 1925, page 421, and in other books on advanced calculus.

$$(25) \qquad \frac{\partial^2 A_n}{\partial \phi^2} + \sin\theta \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial A_n}{\partial \theta}\right) + n(n+1)\sin^2\theta A_n = 0$$

as the *differential equation of the surface spherical harmonics of order n.* We can therefore define the functions in this manner: they are those solutions of (25) which can be represented as homogeneous polynomials of degree $n$ in the arguments $\sin\theta\cos\phi$, $\sin\theta\sin\phi$, $\cos\theta$.

If a surface spherical harmonic $B_n(\theta)$ of the $n$th order is independent of $\phi$, and therefore a function of $\theta$ only, it follows from (25) that it must satisfy the ordinary differential equation

$$(26) \qquad \frac{d}{d\theta}\left(\sin\theta \frac{dB_n}{d\theta}\right) + n(n+1)\sin\theta\, B_n = 0.$$

From (17) the Legendre polynomial $P_n(\cos\alpha)$ is a surface spherical harmonic, where from (8),

$$\cos\alpha = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi').$$

It contains the two variables $\theta$, $\phi$ and the parameters $\theta'$, $\phi'$, which, of course, are also spherical coordinates of a point. If we take $\theta' = 0$, then $\cos\alpha = \cos\theta$, and $P_n(\cos\theta)$ is only dependent on $\theta$. Therefore it must satisfy (26). If we make the substitution $\cos\theta = u$, then (26) takes the form

$$(27) \qquad \frac{d}{du}\left((1-u^2)\frac{dP_n}{du}\right) + n(n+1)P_n$$
$$= (1-u^2)P_n''(u) - 2uP_n'(u) + n(n+1)P_n(u) = 0.$$

Accordingly, $P_n(u)$ must be a solution of (27). This ordinary linear homogeneous differential equation of the second order is known as *Legendre's differential equation.* It has the singular points $u = 1$ and $u = -1$. It possesses, of course, two linearly independent particular solutions. The Legendre polynomial can be taken as one of these; the other is discontinuous at the singular points. This may be shown in

the following manner: let $Q_n(u)$ be a second solution of (27) which is linearly independent of $P_n$. From

$$((1 - u^2)P'_n)' + n(n+1)P_n = 0,$$
$$((1 - u^2)Q'_n)' + n(n+1)Q_n = 0,$$

it follows by multiplication by $Q_n$ and $P_n$ respectively and subsequent subtraction that

$$Q_n((1 - u^2)P'_n)' - P_n((1 - u^2)Q'_n)' = 0,$$

or

$$\{(1 - u^2)(Q_n P'_n - P_n Q'_n)\}' = 0,$$

so that by integration

$$Q_n P'_n - P_n Q'_n = \frac{c}{1 - u^2},$$

where $c$ is a constant. This constant cannot be zero, since if it were zero, we would have

$$\frac{P'_n}{P_n} = \frac{Q'_n}{Q_n}, \text{ so that } P_n(u) = a\,Q_n(u),$$

where $a$ is a constant, and $P_n$ and $Q_n$ would not be linearly independent. Hence it follows that $Q_n$ is discontinuous at $u = 1$ and $u = -1$. The second solution is known as a *Legendre function of the second kind* (see Art. 9).

Since a surface spherical harmonic independent of $\phi$ must satisfy (26) or (27), and since (27) can have only one polynomial solution (except for a constant factor), it follows that the Legendre polynomial $P_n(\cos\theta)$ is the only surface spherical harmonic of order $n$ which is independent of $\phi$.

In order to find further solutions of (25), we seek to satisfy this equation by functions of the form

$$A_n(\theta, \phi) = T(\theta)\,\Phi(\phi) \tag{28}$$

where $T$ and $\Phi$ are respectively functions of $\theta$ and $\phi$ only. Inserting $T\Phi$ in (25), the resulting equation can be easily put in the form

$$\frac{\sin\theta\,(\sin\theta T')' + n\,(n+1)\sin^2\theta\cdot T}{T} = -\frac{\Phi''}{\Phi}.$$

Since the left member of this equation is dependent only on $\theta$, and the right member only on $\phi$, both sides of the equation must be equal to the same constant. We call this constant $\nu^2$, and obtain

$$(29)\qquad \sin\theta(\sin\theta\cdot T')' + [n(n+1)\sin^2\theta - \nu^2]T = 0$$

and

$$\Phi'' + \nu^2\Phi = 0.$$

For $A_n$ to be a surface spherical harmonic, and hence single-valued on the unit-sphere, it is necessary that $\Phi$ have the period $2\pi$, so that $\nu$ must be an integer, and we will therefore make this assumption. Moreover, there is no loss of generality to assume that $\nu$ is not negative. For $\nu = 0$, (29) becomes (26), and we obtain again $P_n(\cos\theta)$.

Let $\cos\theta = u$ and $T(\theta) = w(u)$ in (29), which then takes the form

$$(30)\quad (1-u^2)\,w''(u) - 2uw'(u) + \left(n(n+1) - \frac{\nu^2}{1-u^2}\right)w(u) = 0.$$

This equation has a solution

$$(31)\qquad w(u) = (\sqrt{1-u^2})^\nu P_n^{(\nu)}(u), \qquad P_n^{(\nu)}(u) = \frac{d^\nu P_n(u)}{du^\nu},$$

where $\nu$ is limited to the range $1, 2, 3, \ldots, n$. To prove this, differentiate (27) $\nu$ times, which gives

$$(1-u^2)P_n^{(\nu+2)}(u) - 2u(\nu+1)\,P_n^{(\nu+1)}(u) + [n(n+1) - \nu(\nu+1)]P_n^{(\nu)}(u) = 0.$$

Let the $\nu$th derivative $P_n^{(\nu)}(u) = F(u)$, and this becomes

$$(1-u^2)F''(u) - 2(\nu+1)uF'(u) + [n(n+1) - \nu(\nu+1)]F(u) = 0.$$

If now $(\sqrt{1-u^2})^\nu F(u)$ be substituted in (30), it is easily seen that this equation is satisfied. The notation $P_n^\nu(u)$ is customary for $w(u)$. We write therefore

$$(31^*)\qquad (\sqrt{1-u^2})^\nu P_n^{(\nu)}(u) = P_n^\nu(u), \quad (\nu = 1, 2, \ldots, n).$$

The functions $P_n^\nu(u)$ are called the *associated functions.* One has to distinguish $P_n^\nu(u)$ from the $\nu$th derivative

$$P_n^{(\nu)}(u) = \frac{d^\nu P_n(u)}{du^\nu}.$$

It is easy to see that the functions

$$P_n^\nu(\cos\theta)\cos\nu\phi \quad\text{and}\quad P_n^\nu(\cos\theta)\sin\nu\phi,$$

which certainly satisfy (25), are spherical harmonics of order $n$. For $P_n^{(\nu)}(\cos\theta)$ is a polynomial of the $(n-\nu)$th degree of the form

$$a_0\cos^{n-\nu}\theta + a_2\cos^{n-\nu-2}\theta + a_4\cos^{n-\nu-4}\theta + \ldots$$

and if we multiply the second term of this by

$$\cos^2\theta + \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi = 1,$$

the third by the square of this expression, etc., then $P_n^{(\nu)}(\cos\theta)$ becomes a homogeneous linear combination of degree $n-\nu$ of the three arguments mentioned earlier. Moreover, remembering that $u^2 - 1 = -\sin^2\theta$, we have from well-known trigonometric relations that

$$\sin^\nu\theta\cos\nu\phi = \sin^\nu\theta\left(\cos^\nu\phi - \binom{\nu}{2}\cos^{\nu-2}\phi\sin^2\phi + \ldots\right),$$

$$\sin^\nu\theta\sin\nu\phi = \sin^\nu\theta\left(\binom{\nu}{1}\cos^{\nu-1}\phi\sin\phi - \binom{\nu}{3}\cos^{\nu-3}\phi\sin^3\phi + \ldots\right).$$

These are therefore homogeneous polynomials of degree $\nu$ in the arguments $\sin\theta\sin\phi$, $\sin\theta\cos\phi$. Thus $P_n^\nu(\cos\theta)\begin{cases}\cos\nu\phi\\ \sin\nu\phi\end{cases}$ are homogeneous of degree $n$ in the three arguments.

We have therefore $2n+1$ surface spherical harmonics of order $n$, namely,

$$(32)\qquad P_n(\cos\theta),\ P_n^\nu(\cos\theta)\sin\nu\phi,\ P_n^\nu(\cos\theta)\cos\nu\phi,$$

$$(\nu = 1, 2, 3, \ldots, n).$$

These are evidently linearly independent. We will show later

(Chapter IX) that there are not more than $2n + 1$ such functions; hence every surface spherical harmonic of order $n$ can be expressed as a linear combination of the above functions. (We can call the functions (32) the standard surface spherical harmonics.)

*Remark.* The equation (30) has (like (27)) only one solution, aside from a constant factor, which remains finite at $u = 1$. The reader should prove it.

## Art. 6. Orthogonality

One of the most important properties of the Legendre polynomials is their *orthogonality*. Two functions $f(x)$, $g(x)$ are called *orthogonal* with respect to an interval $a \leq x \leq b$ if the definite integral of their product vanishes, i.e. if

$$\int_a^b f(x)g(x)dx = 0.$$

And a set of a finite or an infinite number of functions $\psi_1(x)$, $\psi_2(x)$, .... is called an orthogonal set, if every function of the set is orthogonal to every other function of the set. If $k_1$, $k_2$, ... is any set of constants, the functions $k_n\psi_n(x) = \phi_n(x)$ evidently form an orthogonal set also. It is well known that the functions

$$\sin nx \ (n = 1, 2, 3, \ldots) \quad \text{and} \quad \cos nx \ (n = 0, 1, 2, 3, \ldots)$$

form an orthogonal set for the interval $0 \leq x \leq 2\pi$, since

$$\int_0^{2\pi} \sin nx \sin mx dx = 0, \quad \int_0^{2\pi} \cos nx \cos mx dx = 0, \text{ for } m \neq n,$$

$$\text{and} \qquad \int_0^{2\pi} \sin nx \cos mx dx = 0.$$

If a function $F(x)$ can be expanded into a uniformly convergent "Fourier series"

$$F(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \ldots + b_1 \sin x + b_2 \sin$$

then the orthogonality property provides a very simple method of evaluating the "Fourier coefficients" $a_n$, $b_n$.

The Legendre polynomials form an orthogonal set of functions for the interval $-1 \leqq u \leqq 1$, i.e.,

$$\int_{-1}^{1} P_n(u)P_m(u)du = 0 \quad (m \neq n). \tag{33}$$

This may be proved from their differential equation (27) and the fact that, being polynomials, the $P_n$ are certainly continuous at the singular points $u = \pm 1$. We find first from (27) that

$$\int_{-1}^{1} P_m \cdot ((1 - u^2)P_n')'du + n(n+1)\int_{-1}^{1} P_m P_n du = 0.$$

Using integration by parts, the first integral becomes

$$\int_{-1}^{1} P_m \cdot ((1 - u^2)P_n')'du = P_m(1 - u^2)P_n' \Big|_{-1}^{1} - \int_{-1}^{1} (1 - u^2)P_m'P_n'du;$$

the first term on the right vanishes on account of the factor $1 - u^2$, since $P_m$ and $P_n'$ remain finite at $u = \pm 1$. Hence

$$-\int_{-1}^{1} (1 - u^2)P_m'P_n'du + n(n+1)\int_{-1}^{1} P_m P_n du = 0.$$

A second similar equation is obtained by permuting $m$ and $n$; then by subtraction,

$$[n(n+1) - m(m+1)]\int_{-1}^{1} P_n P_m du = 0,$$

from which the orthogonality property (33) follows for $m \neq n$.

An orthogonal set of functions $\phi_n(x)$ is said to be *normalized* if each function of the set satisfies the "condition for normality"

$$\int_a^b \phi_n^2 dx = 1.$$

For such a "normal orthogonal system" we have the properties

$$\int_a^b \phi_n \phi_m dx = 0 \quad (m \neq n), \quad \int_a^b \phi_n^2 dx = 1.$$

By means of the quadratic condition for normality, the arbitrary constant $k_n$ in the equation $\phi_n = k_n \psi_n$ is fixed except for sign, since

$$k_n = \pm \cdot \frac{1}{\sqrt{\int_a^b \psi_n^2 dx}}$$

Since

$$\int_0^{2\pi} \sin^2 nx dx = \int_0^{2\pi} \cos^2 nx dx = \pi (n = 1, 2, \ldots), \int_0^{2\pi} dx = 2\pi$$

the functions

$$\frac{1}{\sqrt{2\pi}}, \quad \frac{\sin nx}{\sqrt{\pi}}, \quad \frac{\cos nx}{\sqrt{\pi}},$$

form a normal orthogonal system for the interval $0 \leq x \leq 2\pi$. In order to normalize the Legendre polynomials, we must evaluate $\int_{1-}^{1} P_n^2(u) du$. The value of this integral is

$$\int_{-1}^{1} P_n^2(u) du = \frac{2}{2n+1}. \tag{34}$$

This is easily proved from (12), which for $l = 1$ becomes

$$\sum_{n=0}^{\infty} P_n(u) \rho^n = (1 - 2\rho u + \rho^2)^{-\frac{1}{2}}.$$

Squaring and integrating, we find (remembering the property (33) of orthogonality)

$$\sum_{n=0}^{\infty} \rho^{2n} \int_{-1}^{1} P_n^2 du = \int_{-1}^{1} \frac{du}{1 - 2\rho u + \rho^2}$$
$$= -\frac{1}{2\rho} \left[ \log(1 - 2\rho u + \rho^2) \right]_{-1}^{1}$$

$$= -\frac{1}{2\rho}\left[\log(1-\rho)^2 - \log(1+\rho)^2\right]$$

$$= \frac{1}{\rho}\left[\log(1+\rho) - \log(1-\rho)\right]$$

$$= 2 + \frac{2\rho^2}{3} + \frac{2\rho^4}{5} + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{2}{2n+1}\rho^{2n}.$$

By comparing the coefficients in these two power series, we find (34).

A useful identity is *Rodrigues' formula*

$$P_n(u) = \frac{1}{2^n n!}\frac{d^n(u^2-1)^n}{du^n}. \tag{35}$$

This identity may be obtained by expanding $(u^2-1)^n$ by the binomial theorem, carrying out the differentiation, and comparing the general term of the resulting polynomial of degree $n$ with that of (13).

*Exercise.* Infer from (35) that all roots of $P_n(u)=0$ are real, distinct, and between $-1$ and $+1$.

*Remark.* We mention without proof that equation (35) may be derived from the orthogonality of the Legendre polynomials.

According to (33) and (34), the functions

$$\phi_n(u) = \sqrt{\frac{2n+1}{2}}\,P_n(u) \quad (n = 0, 1, 2, \ldots)$$

form a normal orthogonal system for the interval $-1 \leq u \leq 1$. If a function $F(u)$ can be expanded in a uniformly convergent series of Legendre polynomials

$$F(u) = a_0P_0(u) + a_1P_1(u) + a_2P_2(u) + \ldots + a_nP_n(u) + \ldots$$

then the coefficients $a_n$ can be determined in the Fourier

manner, multiplying the equation by $P_n(u)$ and integrating the series termwise, which is permissible on account of the uniform convergence. On account of the orthogonality property, this gives

$$\int_{-1}^{1} F(u)\cdot P_n(u)du = a_n \int_{-1}^{1} P_n^2(u)du$$

and therefore from (34),

$$a_n = \frac{2n+1}{2}\int_{-1}^{1} F(u)P_n(u)du. \tag{36}$$

We note that the Legendre polynomials are evidently linearly independent since they are each of different degree; and the set of polynomials $P_0, P_1, P_2, \ldots, P_n$ contains polynomials of each degree. We can therefore obviously not only represent the polynomials in powers of $u$, but conversely we can represent $u^n$ as a linear combination of the polynomials $P_0, P_1, \ldots, P_n$:

$$u^n = p_nP_n(u) + p_{n-1}P_{n-1}(u) + \ldots + p_0P_0(u),$$

where $p_n \neq 0$. In fact the solutions of the equations (13) give

$$1 = P_0(u),\ u = P_1(u),\ u^2 = \tfrac{2}{3}P_2(u) + \tfrac{1}{3}P_0(u),$$
$$u^3 = \tfrac{2}{5}P_3(u) + \tfrac{3}{5}P_1(u), \text{ etc.}$$

Hence every polynomial of the $n$th degree can be expressed as a linear combination with constant coefficients of $P_0, P_1, \ldots, P_n$, that is, expanded in a finite series of polynomials $P_n$.

From this it follows that, since $P_n$ is certainly orthogonal to every $P_m$ of lower order, it is orthogonal to all the powers $1,\ u,\ u^2, \ldots,\ u^{n-1}$ and accordingly also to all polynomials in $u$ of degree less than $n$.

The derivative $P_n'(u) = \dfrac{dP_n}{du}$ is a polynomial of degree $n-1$ and can therefore be expressed in the form

$$P_n'(u) = c_1P_{n-1}(u) + \ldots + c_kP_{n-k}(u) + \ldots + c_nP_0(u),$$

where the $c_k$ are constants. The "Fourier coefficients" $c_1, c_2, \ldots, c_n$ are, from (36),

$$c_k = \frac{2(n-k)+1}{2}\int_{-1}^{1} P_{n-k}(u)P_n'(u)\,du$$
$$= \frac{2n-2k+1}{2}\left\{\Big[P_{n-k}P_n\Big]_{-1}^{1} - \int_{-1}^{1} P_nP_{n-k}'\,du\right\}.$$

This last integral is zero, since $P_{n-k}'$ is a polynomial of degree less than $n$, so that

$$c_k = \frac{2n-2k+1}{2}\{1-(-1)^{n-k}(-1)^n\}$$
$$= \begin{cases} 0 \text{ for } k \text{ even,} \\ 2n-2k+1 \text{ for } k \text{ odd.}\end{cases}$$

Therefore

$$P_n'(u) = (2n-1)P_{n-1}(u) + (2n-5)P_{n-3}(u) + (2n-9)P_{n-5}(u) + \ldots.$$

This result is the expansion assumed in (15).

We will now prove: there is only one set of polynomials which (*i*) contains polynomials of each degree, and (*ii*) forms an orthogonal set for the interval $-1 \leq u \leq 1$. (Polynomials are assumed equivalent if one is a multiple of the other.) Let $Q_0, Q_1, Q_2, \ldots,$ be a set of polynomials with the above two properties. Then $Q_n$ must likewise have the property of being orthogonal to every polynomial of degree less than $n$. Let

$$P_n = b_nu^n + b_{n-1}u^{n-1} + \ldots \qquad (b_n \neq 0),$$

and

$$Q_n = d_nu^n + d_{n-1}u^{n-1} + \ldots \qquad (d_n \neq 0).$$

From these, we form

$$R(u) = d_nP_n - b_nQ_n = (d_nb_{n-1} - b_nd_{n-1})u^{n-1} + \ldots.$$

Now $R(u)$ must likewise be orthogonal to every polynomial of degree less than $n$, and hence must be orthogonal to itself. But from $\int_{-1}^{1} R^2(u)du = 0$ it follows that $R(u) \equiv 0$, so that

$P_n$ and $Q_n$ differ from each other only by a constant factor. The Legendre polynomials are therefore fully characterized by the above two properties, aside from constant factors. These can be determined by the requirement that $Q_n(1) = 1$; or else by the normality condition $\int_{-1}^{1} Q_n^2 du = 1$, or the condition that $\int_{-1}^{1} Q_n^2\, du = \frac{2}{2n+1}$, if, in addition, the coefficient of $u^n$ be chosen positive.

The Legendre polynomials can therefore be defined by means of the expansion (12), or as the only continuous solutions of the differential equation (27), or, finally, by the above-described properties.

The property of orthogonality is valid also for the associated Legendre functions, and may be proved in a similar manner. Hence for $m \neq n$,

$$(33^*) \quad \int_{-1}^{1} P_n^\nu(u) P_m^\nu(u) du = \int_{-1}^{1} (1-u^2)^\nu P_n^{(\nu)}(u) P_m^{(\nu)}(u) du = 0.$$

Corresponding to (34), we have

$$(34^*) \qquad \int_{-1}^{1} [P_n^\nu(u)]^2 du = \frac{(n+\nu)!}{(n-\nu)!} \cdot \frac{2}{2n+1}.$$

The concept of orthogonality can be immediately extended to functions of several variables. Two surface spherical harmonics $A_n$, $A_m$ of different order are orthogonal with respect to the surface of the unit sphere, i.e.,

$$(37) \qquad \iint A_m A_n\, dS = 0 \quad (m \neq n),$$

where the integral is over the surface of the unit sphere. This follows, for instance, by setting $u = \rho^n A_n$ and $v = \rho^m A_m$ in Green's formula $\iint_S \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}\right) dS = 0.$ This formula is applicable since $u$, $v$ are regular and harmonic in all finite space.

Since $\dfrac{\partial}{\partial n} = \dfrac{\partial}{\partial \rho}$, we find

$$\iint(\rho^n A_n m\rho^{m-1}A_m - \rho^m A_m n\rho^{n-1}A_n)dS$$
$$= (m - n)\iint A_m A_n dS = 0,$$

since $\rho = 1$, from which the orthogonality property (37) follows. Of course the property (33) may be regarded as a special case of (37), obtained by assuming that $A_n$ and $A_m$ are independent of $\phi$ and making the substitution $\cos\theta = u$.

We will return in Chapter IX to the subject of the expansion of "arbitrary functions" in Legendre polynomials and in general surface harmonics.

*Exercise.* We call two sets of functions $f_0(u) \ldots . f_n(u)$ and $g_0(u) \ldots . g_n(u)$ "linearly equivalent," if each function of either set can be represented as a linear homogeneous expression with constant coefficients of the functions of the other set. Accordingly the sets of Legendre polynomials $P_m(u)$ and of powers $u^m$ $[m = 0, 1, 2, \ldots . n]$ are equivalent. Prove that the coefficient $p_m$ in $u^n = \sum_{m=0}^{n} p_m P_m(u)$ which is given by $p_m = \dfrac{2m+1}{2}\displaystyle\int_{-1}^{1} u^n P_m(u)\, du$ has the value

$$p_m = \left(2m + 1\right)\frac{n(n-1)\ldots\ldots\ldots\ldots(n-m+2)}{(n+m+1)(n+m-1)\ldots(n-m+3)}$$

for $m = n, n-2, \ldots$, and $p_m = 0$ for $m = n-1, n-3, \ldots .$ (Apply (35) and integrate by parts.)

## Art. 7. The Addition Theorem for Spherical Harmonics

We will now prove the following identity:

$$(38) \qquad P_n(\cos\alpha) = P_n(\cos\theta)P_n(\cos\theta')$$
$$+ 2\sum_{\nu=1}^{n}\frac{(n-\nu)!}{(n+\nu)!}P_n^\nu(\cos\theta)P_n^\nu(\cos\theta')\cos\nu(\phi - \phi').$$

Here as in Art. 2, (8)

$$\cos \alpha = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi').$$

Let $\cos \theta = u$, $\cos \theta' = v$, $\phi - \phi' = \psi$, then

$$(8^*) \qquad \cos \alpha = uv + \sqrt{1 - u^2}\sqrt{1 - v^2} \cos \psi$$

and (38) takes the form

$$(38^*) \qquad P_n(\cos \alpha) = P_n(u)P_n(v) + 2 \sum_{\nu=1}^{n} \frac{(n-\nu)!}{(n+\nu)!} P_n^\nu(u) P_n^\nu(v) \cos \nu\psi.$$

The function $P_n$ of the argument $\cos \alpha$, which from (8) is a combination of $\cos \theta$, $\cos \theta'$, $\sin \theta$, $\sin \theta'$, $\cos \psi$, is here expressed in finite form in terms of the function $P_n$ and its derivatives with respect to the arguments $\cos \theta$, $\cos \theta'$ themselves, with powers of $\sin \theta$, $\sin \theta'$ and the functions $\cos \nu\psi$ also entering. The formula (38) is therefore known as the *addition theorem* for spherical harmonics. Since

$$\cos \nu\psi = \cos \nu\phi \cos \nu\phi' + \sin \nu\phi \sin \nu\phi',$$

the addition theorem evidently gives a representation of the spherical harmonic $P_n(\cos \alpha)$ in terms of the $2n + 1$ functions (32). We may consider $P_n(\cos \alpha)$ as a function of the point $P:(\theta, \phi)$ with coefficients which are functions of $Q:(\theta', \phi')$, or *vice versa*.

To prove (38), we note that the function $P_n(\cos \alpha)$ is a polynomial of the $n$th degree of the argument $\cos \psi$. It can therefore be expanded in a finite Fourier's series of cosines of multiples of $\psi$, with coefficients which are functions of $u$ and $v$,

$$(39) \qquad P_n(\cos \alpha) = C_0(u, v) + 2 \sum_{\nu=1}^{n} C_\nu(u, v) \cos \nu\psi.$$

Since $\cos \alpha$ is symmetric in $u$ and $v$, this property must hold for $C_i(u, v)$; for if we permute $u$, $v$ in (39) and subtract the resulting equation from (39), we have

$$C_0(u, v) - C_0(v, u) + 2 \sum (C_\nu(u, v) - C_\nu(v, u)) \cos \nu\psi = 0.$$

This identity can evidently only hold if the coefficients of $\cos \nu\psi$ $(\nu = 0, 1, 2, \ldots, n)$ vanish separately, from which the symmetry conditions $C_\nu(u, v) = C_\nu(v, u)$ follow.

We now compute the Fourier coefficients

$$(40) \quad C_\nu(u,v) = \frac{1}{2\pi}\int_0 P_n(\cos\alpha)\cos \nu\psi d\psi \quad (\nu = 0, 1, \ldots, n).$$

It is easy to see that $C_\nu(u, v)$ as a function of $u$ must satisfy the equation (30); for $P_n(\cos \alpha)$ considered as a function of $\theta$ and $\psi$ satisfies

$$(25) \quad \sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P_n}{\partial\theta}\right) + \frac{\partial^2 P_n}{\partial\psi^2} + n(n+1)\sin^2\theta P_n = 0.$$

On the other hand, by two integrations by parts,

$$\int_0^{2\pi}\frac{\partial^2 P_n}{\partial\psi^2}\cos\nu\psi d\psi = -\nu^2\int_0^{2\pi} P_n\cos\nu\psi d\psi = -2\pi\nu^2 C_\nu.$$

This remains valid also for $\nu = 0$, since

$$\int_0^{2\pi}\frac{\partial^2 P_n}{\partial\psi^2}d\psi = \frac{\partial P_n}{\partial\psi}\Bigg|_{\psi=0}^{\psi=2\pi} = 0$$

($P_n$ being periodic in $\psi$ with the period $2\pi$). If we multiply (25) by $\cos \nu\psi$ and integrate from 0 to $2\pi$, we find for $C_\nu$ as a function of $\theta$ the equation

$$(29) \quad \sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{dC_\nu}{\partial\theta}\right) - \nu^2 C_\nu + n(n+1)\sin^2\theta C_\nu = 0,$$

and hence, as a function of $u = \cos\theta$, actually the equation (30). However, from Art. 5 the equation (30) has only one solution (aside from a constant factor) which remains finite at $u = 1$, namely $\left(\sqrt{1-u^2}\right)^\nu P_n^{(\nu)}(u) = P_n^\nu(u)$. Hence $C_\nu(u, v)$ must be this function multiplied by a factor dependent only on $v$. But since $C_\nu$ is symmetric in $u$, $v$, it follows that for $\nu = 0, 1, 2, \ldots, n$,

$$(41) \qquad C_\nu(u, v) = a_\nu P_n^\nu(u)P_n^\nu(v)$$

where $a_\nu$ is a constant. From (39) and (41) we find that

$$(42)\quad P_n(\cos\alpha) = a_0P_n(u)P_n(v) + 2\sum_{\nu=1} a_\nu P_n^\nu(u)P_n^\nu(v)\cos\nu\psi$$

where the constants $a_0, a_1, \ldots, a_n$ have still to be determined. For this purpose we make the following remark. According to the definition $\cos\theta = u$, $\cos\theta' = v$, we have $|u| \leq 1$, $|v| \leq 1$, if $\theta$ and $\theta'$ are restricted to real values. But we can lift this restriction and can assume that $\theta$, $\theta'$, and so also $u$, $v$, take on arbitrary (real and complex) values. Equation (42) is then a relation between analytic functions. It is known from the theory of analytic functions of a complex variable that, if a relation between such functions holds in any region, however small, it is valid in the whole region in which the appearing functions are defined (principle of the conservation of a relation of functions). Accordingly (42) holds for arbitrary (real and complex) values of $u$ and $v$. We are, therefore, entitled to let $v \to \infty$ and $u \to \infty$.

Remember (compare Art. 2) that $b_n = \dfrac{(2n)!}{2^n(n!)^2}$ is the coefficient of the highest power in the polynomial $P_n(\cos\alpha)$. Divide (42) by $v^n$ and let $v \to \infty$. Then

$$\lim_{v\to\infty}\frac{P_n(\cos\alpha)}{v^n} = b_n\lim_{v\to\infty}\frac{\cos^n\alpha}{v^n} = b_n(u + i\sqrt{1-u^2}\cos\psi)^n,$$

since evidently

$$\lim_{v\to\infty}\frac{\cos\alpha}{v} = \lim_{v\to\infty}\left\{u + \sqrt{1-u^2}\sqrt{\frac{1}{v^2}-1}\cos\psi\right\}$$
$$= u + i\sqrt{1-u^2}\cos\psi.$$

Also

$$\lim_{v\to\infty}\frac{P_n^\nu(v)}{v^n} = \lim_{v\to\infty}\frac{\left(\sqrt{1-v^2}\right)^\nu}{v^\nu}\,\frac{P_n^{(\nu)}(v)}{v^{n-\nu}}$$
$$= i^\nu\lim_{v\to\infty}\frac{P_n^{(\nu)}(v)}{v^{n-\nu}} = i^\nu b_n\frac{n!}{(n-\nu)!},$$

because the highest power of $v$ in $P_n^{(\nu)}(v)$ has the coefficient $b_n \cdot n(n-1)\ldots(n-\nu+1)$. Hence from (42) we get the important formula

$$(u + i\sqrt{1-u^2}\cos\psi)^n = a_0 P_n(u) + 2\sum i^\nu a_\nu \frac{n!}{(n-\nu!)} P_n^\nu(u)\cos\nu\psi$$

or

$$(\cos\theta + i\sin\theta\cos\psi)^n = a_0 P_n(\cos\theta)$$
$$+ 2\sum i^\nu a_\nu \frac{n!}{(n-\nu)!} P_n^\nu(\cos\theta)\cos\nu\psi.$$

If we divide this by $u^n$ and let $u \to \infty$, we get

$$(1-\cos\psi)^n = a_0 b_n + 2\sum i^\nu a_\nu b_n i^\nu \frac{(n!)^2}{[(n-\nu)!]^2}\cos\nu\psi$$
$$= a_0 \frac{(2n)!}{2^n(n!)^2} + 2\sum_{\nu=1}^{n} (-1)^\nu a_\nu \frac{(2n)!}{2^n[(n-\nu)!]^2}\cos\nu\psi.$$

But there is an elementary identity

$$(1-\cos\psi)^n = \frac{(2n)!}{2^n(n!)^2} + 2\sum \frac{(-1)^\nu(2n)!}{2^n(n-\nu)!\,(n+\nu)!}\cos\nu\psi,$$

which may be proved by mathematical induction. By comparison, it is seen that

$$a_\nu = \frac{(n-\nu)!}{(n+\nu)!}$$

for all $\nu$, in particular $a_0 = 1$, which finally proves the addition theorem.

*Exercise.* Find the coefficients in the expansion of $(1-\cos\psi)^n$.

*a*) by the Fourier method, *b*) by mathematical induction, *c*) by setting $1-\cos\psi = 2(\sin\frac{\psi}{2})^2 = -\frac{1}{2}\left(e^{\frac{i\psi}{2}} - e^{\frac{-i\psi}{2}}\right)^2$ and using the binomial theorem.

Moreover, we now have

$$(u + i\sqrt{1-u^2}\cos\psi)^n = P_n(u) + 2\sum i^\nu \frac{n.}{(n+\nu)!} P_n^\nu(u)\cos\nu\psi$$

or

$$(43) \qquad (\cos\theta + i\sin\theta\cos\psi)^n = P_n(\cos\theta) + 2\sum_{\nu=1}^{n} i^\nu \frac{n!}{(n+\nu)!} P_n^\nu(\cos\theta)\cdot\cos\nu\psi.$$

The left side of this important identity is a homogeneous polynomial of the $n$th order in the arguments $\cos\theta$, $\sin\theta\cos\psi$ (the third argument $\sin\theta\sin\psi$ is missing), and is thus a spherical harmonic of order $n$. Equation (43) is therefore the expansion of this spherical harmonic in terms of the functions (32), with the functions $P_n^\nu(\cos\theta)\sin\nu\psi$ not appearing. On the other hand, (43) is the expansion of the function of $\psi$ on the left in a Fourier series (finite). The Fourier coefficients are $P_n(u)$ and associated functions multiplied by constant factors. From this fact, it follows that

$$(44) \qquad P_n(u) = \frac{1}{\pi}\int_0 (u + i\sqrt{1-u^2}\cos\psi)^n\,d\psi,$$

using the usual way of finding Fourier coefficients. This representation of the Legendre polynomial as a definite integral was given by Laplace. In a similar manner, we obtain for the associated functions the integral representation

$$(44^*) \quad P_n^\nu(u) = \frac{(-i)^\nu}{\pi}\frac{(n+\nu)!}{n!}\int_0^\pi (u+i\sqrt{1-u^2}\cos\psi)^n\cos\nu\psi\,d\psi.$$

From the addition theorem itself we get in the same manner

$$(45) \qquad P_n(\cos\theta)P_n(\cos\theta') = \frac{1}{2\pi}\int_0^{2\pi} P_n(\cos\alpha)\,d\psi,$$

and in general

$$(45^*) \quad P_n^\nu(\cos\theta)P_n^\nu(\cos\theta') = \frac{1}{2\pi}\frac{(n+\nu)!}{(n-\nu)!}\int_0^{2\pi} P_n(\cos\alpha)\cos\nu\psi\,d\psi$$

## Art. 8. Expansion of the Potentials at Infinity

We will now expand the potential functions, and also harmonic functions which are regular at infinity, in series convergent in the neighbourhood of infinity. Concerning the definition of regularity at infinity see Chapter II, Art 7.

We begin with the logarithmic potential and apply the transformation of inversion:

$$x' = \frac{x}{x^2+y^2}, \quad y' = \frac{y}{x^2+y^2}, \tag{46}$$

with the inverse

$$x = \frac{x'}{x'^2+y'^2}, \quad y = \frac{y'}{x'^2+y'^2}. \tag{46*}$$

If $\rho = \sqrt{x^2+y^2} \to \infty$, then $\rho' = \sqrt{x'^2+y'^2} \to 0$, so that the neighbourhood of infinity is mapped on the neighbourhood of the origin. For the potential of a simple linear distribution, spread on a curve $C$, viz., $U = \int_C \gamma \log \frac{1}{r}\, ds$, we proceed as follows:

Let
$$\begin{aligned} r^2 &= (x-\xi)^2+(y-\eta)^2 = x^2+y^2-2(x\xi+y\eta)+\xi^2+\eta^2 \\ &= \rho^2-2(x\xi+y\eta)+l^2 \quad (l^2=\xi^2+\eta^2) \\ &= \rho^2\left\{1-2(x'\xi+y'\eta)+\rho'^2l^2\right\}, \end{aligned}$$

and hence

$$\log\frac{1}{r} = \log\frac{1}{\rho} - \frac{1}{2}\log\left\{1-(2x'\xi+2y'\eta-\rho'^2l^2)\right\}.$$

Since
$$-\log(1-z) = z+\frac{z^2}{2}+\frac{z^3}{3}+\dots$$

holds for $|z|<1$, therefore the expansion

$$\begin{aligned} &-\log\left\{1-(2x'\xi+2y'\eta-\rho'^2l^2)\right\} \\ &= (2x'\xi+2y'\eta-\rho'^2l^2)+\frac{(\ \ )^2}{2}+\frac{(\ \ )^3}{3} \\ &= 2(x'\xi+y'\eta)+\dots. \end{aligned}$$

holds for $x'$, $y'$ and hence $\rho'$ sufficiently small. The dots represent terms of the second and higher order in $x'$, $y'$. The separation of terms by the removal of parentheses is justified as in earlier expansions (Art. 1). Hence

$$\log\frac{1}{r} = \log\frac{1}{\rho} + x'\xi + y'\eta + \dots \tag{47}$$

and finally

$$\begin{aligned} U &= \log\frac{1}{\rho}\int_C \gamma ds + x'\int_C \gamma\xi ds + y'\int_C \gamma\eta ds + \dots \\ &= M\log\frac{1}{\rho} + Ax' + By' + \dots \end{aligned}$$

where $A$ and $B$ are constants. Returning to the original coordinates,

$$\begin{aligned} U &= M\log\frac{1}{\rho} + A\frac{x}{x^2+y^2} + B\frac{y}{x^2+y^2} + \dots \\ &= M\log\frac{1}{\rho} + U_1. \end{aligned} \tag{48}$$

This expansion is valid outside a sufficiently large circle about the origin. The series is uniformly convergent and may be differentiated termwise arbitrarily often. The function $U_1$ is obviously harmonic and regular at infinity.

In order to get the potential of a double linear distribution expressed in a series of the above type, we first note that from (47), since $\rho$ is independent of $\xi$, $\eta$,

$$\frac{\partial\left(\log\frac{1}{r}\right)}{\partial n} = x'\frac{\partial\xi}{\partial n} + y'\frac{\partial\eta}{\partial n} + \dots$$

and therefore with constants $C$, $D$,. . .

$$\begin{aligned} U = \int\gamma\frac{\partial\log\left(\frac{1}{r}\right)}{\partial n}ds &= Cx' + Dy' + \dots \\ &= C\frac{x}{x^2+y^2} + D\frac{y}{x^2+y^2} + \dots \end{aligned}$$

The potential of an area distribution

$$\iint_S \sigma \log \frac{1}{r} \, dS$$

is easily seen to lead again to an expansion of the form (48). These expansions are uniformly convergent outside a sufficiently large circle about the origin; they may be differentiated termwise as often as desired.

A harmonic function which is regular at infinity may be expanded in a series convergent in the neighbourhood of infinity. One has to use Chapter III, Eqn. (39), expressing such a harmonic function in terms of line integrals around a closed curve. It is necessary to note here that, since the density of the mass distribution along the curve is $-\frac{1}{2\pi}\frac{\partial u}{\partial n}$, the total mass of this distribution is zero, since

$$M = -\frac{1}{2\pi}\int_C \frac{\partial u}{\partial n} \, ds = 0; \text{ see Chapter III, } (29^*) \cdot$$

Hence it follows immediately that every harmonic function regular at infinity has an expansion of the form

$$u = c + \frac{ax}{x^2 + y^2} + \frac{by}{x^2 + y^2} + \cdots, \tag{49}$$

where $c$, $a$, $b$, . . . are constants.

We will now prove that the inversion (46) transforms a harmonic function $u(x, y)$ into a harmonic function of $x'$, $y'$; i.e., into a function $u\left(\frac{x'}{\rho'^2}, \frac{y'}{\rho'^2}\right)$ satisfying $\frac{\partial^2 u}{\partial x'^2} + \frac{\partial^2 u}{\partial y'^2} = 0$. Since $u$ is expressed in (38), Chapter III, in a form which contains $x$, $y$ only in the form $\log \frac{1}{r}$, it is sufficient to prove the theorem only for this function, or for the function $\log (r^2)$.

Now

$$\log(r^2) = \log\{\rho^2 - 2(x\xi + y\eta) + l^2\}$$
$$= -\log\rho'^2 + \log\{1 - 2(x'\xi + y'\eta) + \rho'^2 l^2\}$$

and setting $\dfrac{\xi}{l^2} = \xi', \dfrac{\eta}{l^2} = \eta', \sqrt{\xi'^2 + \eta'^2} = l'^2$,

$$\log(r^2) = -\log\rho'^2 - \log l'^2 + \log\{\rho'^2 - 2(x'\xi' + y'\eta') + l'^2\}.$$

Evidently, each term on the right in this equation satisfies Laplace's equation in $x'$, $y'$.

From the preceding results we can state the theorem: *A logarithmic potential is regular at infinity if and only if an inversion carries it into a potential regular in the neighbourhood of the origin.*

The expansion of Newtonian potentials in the neighbourhood of infinity is now not difficult. By using the inversion

$$(50)\qquad x' = \frac{x}{\rho^2},\ y' = \frac{y}{\rho^2},\ z' = \frac{z}{\rho^2},\ \rho = \sqrt{x^2 + y^2 + z^2},$$
$$x = \frac{x'}{\rho'^2},\ y = \frac{y'}{\rho'^2},\ z = \frac{z'}{\rho'^2},\ \rho' = \sqrt{x'^2 + y'^2 + z'^2}, \left(\rho' = \frac{1}{\rho}\right),$$

it follows that

$$r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$$
$$= \rho\sqrt{1 - 2(x'\zeta + y'\eta + z'\zeta) + \rho'^2 l^2};\quad (l^2 = \xi^2 + \eta^2 + \zeta^2)$$
$$\frac{1}{r} = \frac{1}{\rho}[1 - 2(x'\xi + y'\eta + z'\zeta) + \rho'^2 l^2]^{-\frac{1}{2}},$$

and hence for sufficiently small $\rho'$,

$$\frac{1}{r} = \frac{1}{\rho}[1 + \tfrac{1}{2}(2x'\xi + 2y'\eta + 2z'\zeta - \rho'^2 l^2) + \dots\dots]$$
$$= \frac{1}{\rho} + \xi\frac{x'}{\rho} + \eta\frac{y'}{\rho} + \zeta\frac{z'}{\rho} + \dots.$$

$$= \frac{1}{\rho} + \xi \frac{x}{\rho^3} + \eta \frac{y}{\rho^3} + \zeta \frac{z}{\rho^3} + \dots ;$$

$$\frac{\partial\left(\frac{1}{r}\right)}{\partial n} = \frac{x}{\rho^3}\frac{\partial \xi}{\partial n} + \frac{y}{\rho^3}\frac{\partial \eta}{\partial n} + \frac{z}{\rho^3}\frac{\partial \zeta}{\partial n} + \dots$$

so that

$$(51) \quad U = \iint_S \frac{\sigma}{r} dS = \frac{M}{\rho} + A_1 \frac{x}{\rho^3} + A_2 \frac{y}{\rho^3} + A_3 \frac{z}{\rho^3} + \dots$$

$$(52) \quad V = \iint_S \mu \frac{\partial\left(\frac{1}{r}\right)}{\partial n} dS = B_1 \frac{x}{\rho^3} + B_2 \frac{y}{\rho^3} + B_3 \frac{z}{\rho^3} + \dots$$

For the potential of a space distribution of mass, the expansion is, of course, of the same form as (51). These series are uniformly convergent outside a sufficiently large sphere about the origin, and may be differentiated termwise there arbitrarily often.

If $u$ is any harmonic function which is regular at infinity, and hence is regular outside some closed surface $S$, then $u$ can be represented (Chapter III, (40)) as the potential of a surface distribution plus the potential of a double layer on $S$. Hence it can be expanded in the form

$$u = \frac{M}{\rho} + \frac{a_1 x}{\rho^3} + \frac{a_2 y}{\rho^3} + \frac{a_3 z}{\rho^3} + \dots = M\rho' + \rho'(a_1 x' + a_2 y' + a_3 z' + \dots),$$

where $\lim \rho u = M$. This series is uniformly convergent and termwise differentiable arbitrarily often outside a sufficiently large sphere.

An essential difference from the behaviour of the logarithmic potential is that the function $u\left(\frac{x'}{\rho'^2}, \frac{y'}{\rho'^2}, \frac{z'}{\rho'^2}\right)$, derived

by the inversion (50) from a potential $u(x, y, z)$, does not satisfy Laplace's equation in $x', y', z'$. On the other hand, $\frac{u}{\rho'}$ is a harmonic function of $x', y', z'$. It is sufficient to prove this for the special case of the potential $\frac{1}{r}$. Now

$$\frac{1}{r\rho'} = \frac{1}{\rho'} \cdot \frac{1}{\sqrt{\left(\frac{x'}{\rho'^2} - \xi\right)^2 + \left(\frac{y'}{\rho'^2} - \eta\right)^2 + \left(\frac{z'}{\rho'^2} - \zeta\right)^2}}$$

$$= \frac{1}{\sqrt{1 - 2(x'\xi + y'\eta + z'\zeta) + l^2\rho'^2}}, \quad (l^2 = \xi^2 + \eta^2 + \zeta^2).$$

Let $\frac{\xi}{l^2} = \xi', \ \frac{\eta}{l^2} = \eta', \ \frac{\zeta}{l^2} = \zeta', \sqrt{\xi'^2 + \eta'^2 + \zeta'^2} = l' = \frac{1}{l}$,

then

$$\frac{1}{r\rho'} = \frac{l'}{\sqrt{(x' - \xi')^2 + (y' - \eta')^2 + (z' - \zeta')^2}}$$

and this function is evidently harmonic in $x', y', z'$.

From the foregoing work it may be seen that: *A harmonic function $u(x, y, z)$ is regular at infinity if and only if the corresponding function*

$$v\,(x', y', z') = \frac{1}{\rho'}\, u\left(\frac{x'}{\rho'^2}, \frac{y'}{\rho'^2}, \frac{z'}{\rho'^2}\right),$$

*harmonic in $x', y', z'$, is regular at the origin.*

We will now discuss the position of extrema of a harmonic function regular at infinity. The theorem that a harmonic function can have no maximum or minimum in the interior of its region of regularity does not in general hold for infinite regions of regularity. This can be seen from the simple example $u = \frac{M}{\rho}$, since this function has a minimum at infinity. From

the above series expansions, it is evident that every harmonic function regular at infinity, whose mass $M \neq 0$, has a minimum at infinity (or maximum if $M < 0$); for it is evident that the potential has the same sign as $M$ outside a sufficiently large sphere, and vanishes at infinity. On the other hand, it is true that: *A harmonic function regular at infinity with a mass $M=0$ cannot have an extremum at infinity.* This may be proved as follows: If $u(x, y, z)$ has an extremum, say a minimum, at infinity (of course $u \to 0$ as $\rho \to \infty$), then the function $u\left(\frac{x'}{\rho'^2}, \frac{y'}{\rho'^2}, \frac{z'}{\rho'^2}\right)$ must have a minimum 0 at the origin in the $(x', y', z')$-space; that is, $u$ is 0 at the origin and is positive in the neighbourhood of the origin. Now not merely $u$, but also $\frac{u}{\rho'}$ vanishes at the origin of $(x', y', z')$-space, for it was assumed that $M = \lim\limits_{\rho \to \infty} \rho u = \lim\limits_{\rho' \to 0} \frac{u}{\rho'} = 0$. But since $\frac{u}{\rho'}$ is positive in the neighbourhood of the origin, it must have a minimum there; this, however, is impossible since $\frac{u}{\rho'}$ is a regular harmonic function in the neighbourhood of the origin. Hence the theorem is proved.

Just as $\frac{1}{r}$ was expanded in the neighbourhood of the origin in positive powers of $\rho$, it can be expanded in negative powers in the neighbourhood of infinity. We obtain, since $r$ is symmetric in $P$ and $Q$,

$$(53) \quad \frac{1}{r} = \sum_{n=0}^{\infty} P_n(u) \frac{l^n}{\rho^{n+1}} = \frac{1}{\rho} + P_1(u) \frac{l}{\rho^2} + P_2(u) \frac{l^2}{\rho^3} + \dots$$

and this series is absolutely convergent for $\rho > l$, and is uniformly convergent for every region which is entirely exterior to the sphere of radius $l$ about the origin. The expansion for the

derivatives of $\frac{1}{r}$ are obtained by termwise differentiation of (53).

The general term

$$P_n(u)\frac{l^n}{\rho^{n+1}} = P_n(u) \cdot \rho^n \frac{l^n}{\rho^{2n+1}}$$

is equal to a spherical harmonic of order $n$, divided by $\rho^{2n+1}$. Such a function is harmonic. For if $w(x, y, z)$ is an arbitrary harmonic function, then $v(x', y', z') = \frac{1}{\rho'} w\left(\frac{x'}{\rho'^2}, \frac{y'}{\rho'^2}, \frac{z'}{\rho'^2}\right)$ is likewise a harmonic function of $x', y', z'$. If $w$ is a homogeneous polynomial of degree $n$ in $x, y, z$ then it follows that

$$v(x', y', z') = \frac{1}{\rho'} \cdot \frac{1}{\rho'^{2n}} w(x', y', z') = \frac{1}{\rho'^{2n+1}} w(x', y', z').$$

This function is therefore harmonic in $x', y', z'$. But since it is immaterial how the variables are designated, it follows that

$$\frac{w(x, y, z)}{\rho^{2n+1}}$$

is harmonic in $x, y, z$.

From the above expansions of $\frac{1}{r}$ and that of its derivative

$$\frac{\partial\left(\frac{1}{r}\right)}{\partial n},$$

the expansion of the three Newtonian potentials,[5] in the neighbourhood of infinity is obtained by multiplying by the density (or moment of a double layer) and integrating termwise. Each of these potentials may therefore be developed in a series whose general term is a spherical harmonic of order $n$ divided by $\rho^{2n+1}$. These are absolutely and uniformly

[5]The potential of a body, of a surface, and of a double layer.

convergent in the region outside a sphere which contains the distribution in its interior, and may be differentiated termwise as often as desired. Similar expansions are valid for any function harmonic and regular in the neighbourhood of infinity.

*Exercise.* Prove the following theorem: *A harmonic function $u(x, y)$ which is regular at infinity can have no extremum at infinity.* Note that in the two-dimensional (logarithmic) case a harmonic function is regular at infinity if and only if its mass $M = 0$. The mass is defined to be the limit

$$\lim_{\rho \to \infty} \frac{u(x, y)}{\log \frac{1}{\rho}} = M.$$

### Art. 9. Exercises concerning Legendre Functions

We want to apply the theory of analytic functions of a complex variable to Legendre functions. Our starting point is Cauchy's integral theorem.

$$g(u) = \frac{1}{2\pi i} \int_C \frac{g(z)}{z - u} dz \tag{54}$$

where $g(u)$ is an analytic function of $u$ and $C$ is a closed curve, which lies entirely in the region of regularity of $g(u)$ and encircles the point $u$ in the positive sense (counterclockwise). The derivatives of $g(u)$ are given by

$$\frac{d^n g(u)}{du^n} = g^{(n)}(u) = \frac{n!}{2\pi i} \int_C \frac{g(z)}{(z - u)^{n+1}} dz \quad [n = 1, 2, \ldots]. \tag{55}$$

Put $\dfrac{(u^2 - 1)^n}{2^n} = g(u)$, use (35) and (55) and show that

$$P_n(u) = \frac{1}{2\pi i} \cdot \frac{1}{2^n} \int_C \frac{(z^2 - 1)^n}{(z - u)^{n+1}} dz. \tag{56}$$

From this we can derive again Legendre's differential equation (27). Show that

$$(1 - u^2)P_n''(u) - 2uP_n'(u) + n(n+1)P_n(u) = \frac{n+1}{2^{n+1}\pi i}\int_C \frac{d}{dz}\frac{(z^2-1)^{n+1}}{(z-u)^{n+2}}\,dz$$

and notice that the integral vanishes, since $C$ is a closed curve.

The integral in (56) could be used to extend the definition of $P_n(u)$ to the case where $n$ is not a positive integer, but we shall not study this problem.

The *Legendre function of the second kind* $Q_n(u)$ is defined by

$$Q_n(u) = \frac{1}{2^{n+1}}\int_{-1}^{+1}\frac{(1-z^2)^n}{(u-z)^{n+1}}\,dz \tag{57}$$

for all finite points $u$ except the points of the linear segment $-1 \leqq u \leqq 1$.

Show that $Q_n(u)$ satisfies equation (27), since obviously

$$\int_{-1}^{+1}\frac{d}{dz}\frac{(z^2-1)^{n+1}}{(z-u)^{n+2}}\,dz = 0.$$

Show that

$$\begin{aligned} Q_0(u) &= \frac{1}{2}\log\frac{u+1}{u-1} \\ Q_1(u) &= \frac{u}{2}\log\frac{u+1}{u-1} - 1. \end{aligned} \tag{58}$$

After we have given a method to derive (27) from (56), it is not difficult to find again the recursion-formulas of Art. 2 on the basis of (56). Take, for instance, the expression on the left side of the formula ($C$) of Art. 2 and show that it can be represented as the integral of a differential quotient.

We can prove likewise on the basis of the definition (57), that $Q_n(u)$ satisfies the same recursion-formulas as $P_n(u)$.

Derive from the validity of the same recursion-formula ($A$) of Art. 2 for $P_n(u)$ and $Q_n(u)$ that

$$Q_n(u) = \tfrac{1}{2}\,P_n(u)\log\frac{u+1}{u-1} + p_{n-1}(u), \tag{58*}$$

where $p_{n-1}(u)$ means a polynomial of degree $n - 1$. Apply mathematical induction. (In the case $n = 0$ one has to put 0 instead of $p_{n-1}(u)$).

According to (58*), $Q_n(u)$ is defined for all finite points $u$, except for the two points $u = 1$ and $u = -1$. They are singular points of $Q_n(u)$.

# CHAPTER V

## BEHAVIOUR OF THE POTENTIAL AT POINTS OF THE MASS

### Art. 1. Auxiliary Considerations

It has been seen that all potentials are regular analytic functions at all points outside the masses. We will now consider how the potential behaves when the field-point $P$ approaches the masses causing the field, or moves in the interior of these masses. We will consider this problem in detail only for Newtonian potentials; the corresponding theorems for logarithmic potential, which may be proved in a similar manner, will be merely stated or inferred.

The potential of a space distribution,

$$\iiint\limits_V \frac{\tau}{r}\, dV,$$

exists also when the field-point $P$ is in the mass, i.e. in the region $V + S$. Let the boundary $S$ be a closed surface, with continuously changing normal. Introduce the spherical coordinates $r$, $\theta$, $\phi$ with $P$ as origin, and the integral for the potential becomes

$$\iiint\limits_V \tau r \sin \theta dr d\theta d\phi.$$

Thus the potential integral is changed by this transformation of coordinates from an improper integral to an absolutely convergent one (proper integral). This is also true for the integrals giving the force field, as for example

$$\iiint_V \frac{\tau(\xi - x)}{r^3}\, dV = \iiint_V \tau \frac{\xi - x}{r} \sin \theta dr d\theta d\phi,$$

remembering that $\dfrac{|\xi - x|}{r} \leq 1.$

In order to penetrate more deeply into the subject, we must get inequalities for certain integrals.

We seek an upper bound for the integral

$$\iiint_V \frac{1}{r}\, dV$$

which is to be independent of the position of $P$ and dependent only on the volume $V$ of the region $V$. Let $K$ be a sphere about $P$ as centre, having the same volume $V$. It has therefore the radius $l = \left(\dfrac{3V}{4\pi}\right)^{\frac{1}{3}}$, since $V = \dfrac{4\pi l^3}{3}$. Then we can prove that

(1) $$\iiint_V \; r \quad [illegible] \quad 4\pi$$

irrespective of whether $P$ is inside or outside $V$ or on $S$. For,

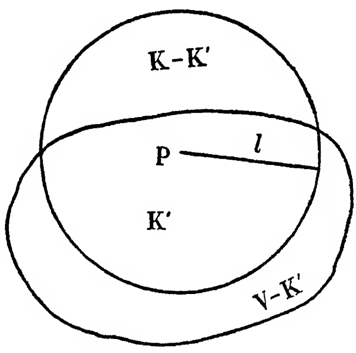


FIG. 7

let $K'$ be the part of the region $V$ which is inside the sphere $K$, and also let $K'$ stand for the volume of this region. Then

$$\iiint\limits_{K} \frac{1}{r} dV = \iiint\limits_{K'} \frac{dV}{r} + \iiint\limits_{K-K'} \frac{dV}{r}$$

$$\geq \iiint\limits_{K'} \frac{dV}{r} + \frac{1}{l}\iiint\limits_{K-K'} dV = \iiint\limits_{K'} \frac{dV}{r} + \frac{K-K'}{l} \quad (K=V)$$

since $r \leq l$ in the region $K-K'$. Also

$$\iiint\limits_{V} \frac{dV}{r} = \iiint\limits_{K'} \frac{dV}{r} + \iiint\limits_{V-K'} \frac{dV}{r}$$

$$\leq \iiint\limits_{K'} \frac{dV}{r} + \frac{1}{l} \iiint\limits_{V-K'} dV = \iiint\limits_{K'} \frac{dV}{r} + \frac{V-K'}{l}$$

since $r \geq l$ in the region $V - K'$. Combining these inequalities, we find

$$\iiint\limits_{V} \frac{dV}{r} \leq \iiint\limits_{K} \frac{dV}{r} = \int_0^l rdr \int_0^{2\pi}\int_0^{\pi} \sin\theta d\theta d\phi =$$

$$\frac{l^2}{2} \cdot 2 \cdot 2\pi = 2\pi\left(\frac{3V}{4\pi}\right)^{\frac{2}{3}}.$$

Another inequality which holds for all positions of the point $P$ is

(2)
$$\iiint\limits_{V} \frac{dV}{r^2} \leq 4\pi\left(\frac{3V}{4\pi}\right)^{\frac{1}{3}}.$$

This is proved in a similar manner, using

$$\iiint\limits_{K} \frac{dV}{r^2} = \iiint \sin\theta dr d\theta d\phi = 4\pi l = 4\pi\left(\frac{3V}{4\pi}\right)^{\frac{1}{3}}.$$

In the plane, it is found in a similar way that

$$\iint_S \frac{dS}{r} \leq 2\pi\left(\frac{S}{\pi}\right)^{\frac{1}{2}}, \tag{3}$$

where the quantity $S$ is the area of the plane region $S$. And similarly, the more general inequality, valid for $0 < \sigma < 2$,

$$\iint_S \frac{dS}{r^{2-\sigma}} \leq \frac{2\pi}{\sigma}\left(\frac{S}{\pi}\right)^{\sigma/2} \tag{3*}$$

which contains (3) as the special case $\sigma = 1$.

Finally, we consider the integral

$$\iint_S \frac{dS}{r}$$

over any bounded surface $S$ with continuously changing normal. By a rotation of the coordinate system, this may be taken so that the normal at a particular point $Q$ of $S$ is parallel to the $z$-axis, and hence $\cos(n, z) = 1$ there. For any pre-assigned constant $c$, satisfying the inequality $0 < c < 1$, on account of the continuity of the normal there is a region[1] about $Q$ such that for all points of this region $\cos(n, z) \geq c$. We assume that the surface $S$ is small enough so that this inequality holds over it. Then for this surface

$$\iint_S \frac{dS}{r} \leq \frac{2\pi}{c}\left(\frac{\omega}{\pi}\right)^{\frac{1}{2}} \tag{4}$$

where $\omega$ is the area of the projection $T$ of $S$ on the $(x, y)$-plane. For, let $r'$ be the projection of $r$ on the $(x, y)$-plane and $d\omega$ be the projection of $dS$. Then $\dfrac{d\omega}{dS} = \cos(n, z)$, so that

$$dS = \frac{d\omega}{\cos(n, z)} \leq \frac{d\omega}{c}, \text{ and also } r' \leq r;$$

[1]The portion of $S$ cut off by a sphere about $Q$ is considered here.

hence

$$\iint\limits_S \frac{dS}{r} \leq \frac{1}{c} \iint\limits_T \frac{d\omega}{r'} \leq \frac{2\pi}{c}\left(\frac{\omega}{\pi}\right)^{\frac{1}{2}} \quad \text{from (3).}$$

The more general inequality

$$\text{(4*)} \qquad \iint\limits_S \frac{dS}{r^{2-\sigma}} \leq \frac{2\pi}{c\sigma}\left(\frac{\omega}{\pi}\right)^{\sigma/2} \qquad (0 < \sigma < 2)$$

is proved in a similar manner.

## Art. 2. Continuity of the Potential of a Body and of its First Derivatives

*The potential*

$$U = \iiint\limits_V \frac{\tau dV}{r}$$

*is continuous when the field-point $P$ lies inside or on the boundary of $V$.* To prove this, it must be shown that, for any preassigned $\epsilon > 0$, the inequality

$$\text{(5)} \qquad |U_P - U_{P'}| < \epsilon$$

holds for the distance $\overline{PP'}$ sufficiently small, where $U_P$ and $U_{P'}$ are the values of $U$ at $P$ and $P'$. First let $P$ lie inside $V$ (not on $S$). Let $K$ be a sphere about $P$ of radius $\delta$ small enough so that $\left|\iiint\limits_K \frac{\tau}{r} dV\right| < \frac{\epsilon}{3}$, and so that $K$ lies inside $V$; if be the maximum of $|\tau|$ in $V + S$, then this can be accomplished if $\delta$ satisfies the inequality $2\pi N\delta^2 < \frac{\epsilon}{3}$, since from (1),

$$\left|\iiint \frac{\tau}{r} dV\right| \leq N \cdot 2\pi\delta^2.$$

After $\delta$ has been chosen in this way, and thus the sphere $K$ fixed, let

$$U = \iiint\limits_{K} \frac{\tau}{r}\, dV + \iiint\limits_{V-K} \frac{\tau}{r}\, dV = U^{(1)} + U^{(2)};$$

then $|U_P^{(1)}| < \frac{\epsilon}{3}$, $|U_{P'}^{(1)}| < \frac{\epsilon}{3}$ so that no matter where $P'$ is located,

$$(6) \qquad |U_P^{(1)} - U_{P'}^{(1)}| < \frac{2\epsilon}{3}.$$

But for the integral $U^{(2)}$, $P$ is an exterior point, so that the continuity of this integral is trivial, and hence

$$(7) \qquad |U_P^{(2)} - U_{P'}^{(2)}| < \frac{\epsilon}{3}$$

for $\overline{PP'}$ sufficiently small. From (6) and (7) together we have (5).

If $P$ lies on the surface $S$, then the above proof can be repeated, the only difference being that $K$ is to be that part of the sphere inside $V$.

The proof of the continuity (for points in $V$) of

$$X = \iiint\limits_{V} \tau \frac{\partial\left(\frac{1}{r}\right)}{\partial x}\, dV = \iiint\limits_{V} \tau \frac{\xi - x}{r^3}\, dV$$

and of $Y$ and $Z$ is made in a similar manner, making use of the inequality (2) instead of (1). From this it does not follow, however, that the equation $(X, Y, Z) = \operatorname{grad} U$ must also hold inside the mass. Rather, it is still necessary to prove that differentiation under the integral sign is permissible. Enclose the point $P$ in a small circular cylinder with axis parallel to the $x$-axis, of radius $\delta$, and length an arbitrary constant, say 1. Let $P$ be at the mid-point of the axis of the cylinder. Let $V_1$ be that part of $V$ which is in the cylinder, and $V_2$ be the remainder of $V$; let the integrals over $V_1$ and $V_2$ be respec-

tively $U^{(1)}$, $X^{(1)}$ and $U^{(2)}$, $X^{(2)}$. Then, since $P$ is an exterior point of $V_2$,

$$\frac{\partial U^{(2)}}{\partial x} = X^{(2)},$$

and this equation is valid for $P$ itself and for points inside the cylinder, on its axis. Let $P'$ $(x', y, z)$ be such a point; then by integration

$$U^{(2)}(x, y, z) - U^{(2)}(x', y, z) = \int_{x^1}^{x} X^{(2)}(t, y, z)\, dt.$$

Now it follows from (1) and (2) that $U^{(1)} \to 0$ and $X^{(1)} \to 0$ as $\delta \to 0$, and this convergence is uniform for all positions of the point in the neighbourhood of $P$. Hence, if we pass to the limit, since $X^{(2)} \to X$ uniformly, we may interchange the order of integration and passage to the limit, so that

$$U(x, y, z) - U(x', y, z) = \int_{x^1}^{x} X(t, y, z)\, dt.$$

From this it follows immediately that

$$\frac{\partial U}{\partial x} = X. \tag{8}$$

Similarly the corresponding equations for $Y$ and $Z$ are proved. Hence: *The first derivatives of the potential $U$ of a space distribution are continuous in the whole of space, and $F = grad\ U$ everywhere.*

## Art. 3. Poisson's Equation

In the preceding article, we showed that $U$ and its first derivatives are continuous for all of space, in the case of a space distribution, under the assumption that the density is bounded and integrable and that $S$ has a continuous normal. We will now discuss the second derivatives at points in the interior of $V$. We make the assumption that the density $\tau$ has continuous first derivatives as well as being bounded,

and still assume that $S$ has a continuous normal. When $P$ is outside $V$, we can write

$$\frac{\partial U}{\partial x} = -\iiint_V \tau \frac{\partial\left(\frac{1}{r}\right)}{\partial \xi} dV = -\iiint_V \frac{\partial}{\partial \xi}\left(\frac{\tau}{r}\right) dV + \iiint_V \frac{1}{r}\frac{\partial \tau}{\partial \xi} dV$$

$$= -\iint_S \frac{\tau}{r} \cos(n,x) dS + \iiint_V \frac{1}{r}\frac{\partial \tau}{\partial \xi} dV$$

by use of the divergence theorem. But this transformation can be shown to be valid also for $P$ inside $V$ since all the integrals are convergent for such a point.

Hence for $P$ inside $V$

$$\frac{\partial U}{\partial x} = \iiint_V \frac{1}{r}\frac{\partial \tau}{\partial \xi} dV - \iint_S \frac{\tau \cos(n,x)}{r} dS = A(x,y,z) + B(x,y,z). \tag{9}$$

Here $A$ is the potential of a space distribution of density $\dfrac{\partial \tau}{\partial \xi}$ and $B$ is the potential due to a surface distribution of density $-\tau\cos(n, x)$ on $S$. Thus both $A$ and $B$ have continuous first derivatives everywhere except on $S$ itself. Hence $U$ nas continuous second derivatives. These are given by

$$(10)\quad \begin{cases} \dfrac{\partial^2 U}{\partial x^2} = \dfrac{\partial A}{\partial x} + \dfrac{\partial B}{\partial x} \\ \qquad = \displaystyle\iiint_V \frac{\partial \tau}{\partial \xi}\frac{\partial\left(\frac{1}{r}\right)}{\partial x} dV - \iint_S \tau\cos(n,x)\frac{\partial\left(\frac{1}{r}\right)}{\partial x} dS, \\ \dfrac{\partial^2 U}{\partial x \partial y} = \displaystyle\iiint_V \frac{\partial \tau}{\partial \xi}\frac{\partial\left(\frac{1}{r}\right)}{\partial y} dV - \iint_S \tau\cos(n,x)\frac{\partial\left(\frac{1}{r}\right)}{\partial y} dS, \\ \text{etc.} \end{cases}$$

*The second derivatives of the potential of a space distribution are continuous at points both outside and inside the mass, but not for passage through the surface S bounding the mass.* They are indeed not defined at points of $S$. If the hypotheses on $\tau$ are met only in a sub-region $V'$ of $V$ having a boundary with continuous normal, we can divide the potential into two parts

$$U = \iiint\limits_{V'} \frac{\tau}{r}\, dV + \iiint\limits_{V-V'} \frac{\tau}{r}\, dV = U' + U''.$$

If we restrict $P$ to motion in $V'$, then $U'$ has continuous second derivatives from (10) while $U''$ has continuous second derivatives because $P$ is outside its region of integration; hence $U$ has continuous second derivatives in $V'$, given by (10).

From the formulas (10), we find

$$\nabla^2 U = -\iiint\limits_{V} \nabla\tau \cdot \nabla\left(\frac{1}{r}\right) dV + \iint\limits_{S} \tau \frac{\partial\left(\frac{1}{r}\right)}{\partial n}\, dS,$$

$$\left[\nabla = \left(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta}, \frac{\partial}{\partial \zeta}\right)\right]$$

since

$$\frac{\partial r}{\partial x} = -\frac{\partial r}{\partial \xi}, \text{ and } \frac{\partial\left(\frac{1}{r}\right)}{\partial n} = \frac{\partial\left(\frac{1}{r}\right)}{\partial \xi}\frac{\partial \xi}{\partial n} + \cdots$$

$$= \frac{\partial\left(\frac{1}{r}\right)}{\partial \xi} \cos(n, x) + \cdots.$$

From the formula (36), Chapter 3, with $u = \tau$, we have at once

$$\nabla^2 U = -4\pi\tau_P, \tag{11}$$

which is *Poisson's equation.*

Poisson's equation is valid for all points $P$ of $V$ (or for all

points in any sub-region of $V$ in which the density $\tau$ has continuous first derivatives).

*The potential of a space distribution therefore satisfies Poisson's equation at all points inside the mass, and satisfies Laplace's equation at all outside points. Neither of these equations is valid on the boundary.*

Laplace's equation can be regarded as a special case of Poisson's equation, since it is obtained from Poisson's equation by setting $\tau = 0$. Correspondingly, the integral for the potential can be considered as extended over all of space, with the density $\tau = 0$ at all points where there are no masses. Then Poisson's equation is valid in all of space except those points where $\tau$ is discontinuous, or its first derivatives are discontinuous, and hence certainly with the exception of the points of $S$ since $\tau$ has a finite discontinuity (a jump) on $S$.

## Art. 4. Continuity of Potential of Surface Distribution

We can prove that: *The potential of a surface distribution*

$$U = \iint \frac{\sigma}{r} dS$$

*is also continuous at the points $P$ of $S$.* Since $\sigma$ is assumed to be bounded, the absolute convergence of the integral $U$ for a point $P$ on $S$ follows from (4). In order to prove the continuity of $U$, we proceed in a manner similar to the proof in Art. 2. It is sufficient to sketch the proof. We cut out from $S$ a small neighbourhood $S_1$ of $P$, and call the remainder of the surface $S_2$. The corresponding integrals are denoted by $U^{(1)}$ and $U^{(2)}$, then $U = U^{(1)} + U^{(2)}$. By the use of (4), we find that $|U^{(1)}| < \frac{\epsilon}{3}$ when $S_1$ is sufficiently small, and this holds uniformly for all $P$. On the other hand, $P$ is an exterior point for $U^{(2)}$.

## Art. 5. Discontinuity of Potential of a Double Layer

In contrast to the potential of a surface distribution, the potential of a double layer is discontinuous at the surface carrying the distribution. We will prove that: *The potential of a double layer,*

$$(12) \qquad U = \iint_S \mu \frac{\partial\left(\frac{1}{r}\right)}{\partial n}\, dS = \iint_S \frac{\mu \cos(r, n)}{r^2}\, dS,$$

*has a jump in value*

$$(13) \qquad U_+ - U_- =$$

*on passing through $S$ at $P$ in the direction of* $\boldsymbol{n}$. We recall that $\boldsymbol{r}$ is directed from the integration point $Q:(\xi, \eta, \zeta)$ to the field-point $P:(x, y, z)$; for simplicity we make the substitution $\psi$ for the angle $(r, n)$, so that

$$(12^*) \qquad U = \iint_S \frac{\mu \cos \psi}{r^2}\, dS.$$

We first assume that $S$ is a closed surface and consider the special case of a double layer of constant moment $\mu = 1$, and hence study the integral

$$(14) \qquad \Omega = \iint_S \frac{\partial\left(\frac{1}{r}\right)}{\partial n}\, dS = \iint_S \frac{\cos \psi}{r^2}\, dS.$$

The discontinuity of $\Omega$ is easily recognized when the geometric meaning of the integrand $\frac{\cos \psi}{r^2}$ is brought out. We consider two spheres $K_1$ and $K$ of radius 1 and $r$ about $P$ as centre, and let $d\omega$ and $dA$ be the areas cut out on these by the cone which the area element $dS$ subtends at $P$. Then it is evident that $dA = |\cos \psi|\, dS$ and $d\omega = \frac{dA}{r^2}$, so that

$$\frac{|\cos\iota\,|\,dS}{\;}, \text{ or } \frac{\cos\psi dS}{\;} = \pm\, d\omega$$

where the upper or lower sign is used according as the angle between the normal and the direction $QP$ is acute or obtuse; thus the integrand is the element of solid angle subtended at $P$ by $dS$. Hence it is easily seen that assuming $P$ inside $S$

$$(15_1) \qquad \Omega = \iint\limits_S \frac{\cos\psi dS}{r^2} = -\iint\limits_S d\omega = -\,4\pi,$$

if the normal is outward from the closed surface $S$. For, any half-ray directed inward toward $P$ must pierce $S$ an odd number of times (remember that the surface is closed), as it must eventually pass from outside to inside; at the points where it enters the surface $S$, the integrand of (14) is the negative of the element of solid angle which $dS$ subtends at $P$. (Solid angle of a cone is the area it cuts out on a unit sphere about its vertex.) At any point where a half-ray toward $P$ leaves $S$, the integrand reduces to $+d\omega$. If a ray enters and leaves $S$ several times before reaching $P$, this gives a contribution $-d\omega + d\omega - \ldots + d\omega - d\omega = -d\omega$ to the integral (14), so that this integral has the value of minus the entire surface of the unit sphere.

If $P$ lies outside $S$, then (14) has the value

$$(15_2) \qquad \Omega = 0,$$

since any half-ray toward $P$ must cut $S$ an even number of times if at all. Finally, if $P$ is on $S$, at an ordinary point of $S$, we have

$$(15_3) \qquad \Omega = -2\pi$$

since the elements $d\omega$ cover half of the unit sphere about $P$ (one side of the plane tangent to $S$).

If we permit $S$ to have singular points (finite in number) such as the vertices of cones, then when $P$ is such a point,

$$(15^*_3) \qquad \Omega = -\gamma,$$

where $\gamma$ is the solid angle of the tangent cone to $S$ at $P$. Of course $(15_3)$ is a special case of $(15^*_3)$. The equations (15) show the discontinuity of (14) on passing through $S$.

We return now to $U = \iint\limits_S \frac{\mu \cos \psi}{r^2}\, dS$; let $\mu_A$ be the moment of the double layer at a definite point $A$ of $S$; write $U$ in the form

$$U = \mu_A \iint \frac{\cos \psi}{r^2} dS + \iint \frac{(\mu - \mu_A) \cos \psi}{r^2} dS$$

or

$$(16) \qquad U - \mu_A \Omega = \iint\limits_S (\mu - \mu_A) \frac{\cos \psi}{r^2} dS = f.$$

We will show that $f$ remains continuous when $P$ passes through $S$ at the point $A$. Assuming this for the present, and letting the subscripts $-, A, +$ denote the values of functions on approaching $A$ from the interior, at $A$, and on approaching $A$ from the exterior (positive side), we have

$$\Omega_- = -4\pi, \quad \Omega_A = \begin{cases} -2\pi \\ -\gamma \end{cases}, \quad \Omega_+ = 0, \quad f_- = f_A = f_+.$$

Hence if $A$ is an ordinary point on the surface $S$,

$$U_- + 4\pi\mu_A = U_+ = U_A + 2\pi\mu_A$$

or

$$(17) \qquad U_- = U_A - 2\pi\mu_A, \quad U_+ = U_A + 2\pi\mu_A.$$

These equations can also be written in the form

$$(18) \qquad U_+ - U_- = 4\pi\mu_A, \quad \frac{U_+ + U_-}{2} = U_A.$$

If $A$ is a singular point of $S$, then

$$U_- + 4\pi\mu_A = U_+ = U_A + \gamma\mu_A$$

or

(17*) $U_+ - U_- = 4\pi\mu_A,\ U_- = U_A - (4\pi - \gamma)\mu_A,\ U_+ = U_A + \gamma\mu_A.$

If the surface is not closed, then we can enlarge $S$ to the closed surface $S'$, extending the definition of $\mu$ in any arbitrary continuous manner over the added portion of $S'$. The potential $U'$ obtained by integrating over the extended surface $S'$ satisfies the conditions (17) or (17*); but the potential $U' - U$ obtained by integrating over $S' - S$ is continuous at $A$ (if $A$ doesn't lie on the boundary of the open surface $S$), since $A$ is not on this surface. Hence the validity of (17), (17*), and (18) follows for the potential $U$ of a distribution of moment $\mu$ over the open surface $S$ (if $A$ doesn't lie on the boundary of $S$).

We will now complete the proof by showing the continuity of

$$f = \iint\limits_S (\mu - \mu_A)\frac{\cos\psi}{r^2}\,dS$$

at the point $A$. Surround $A$ by a small sphere $K$ of radius $\delta$, which cuts off the portion $S_1$ of $S$; let $S_2$ be the remainder of $S$. Let $f_1$ and $f_2$ be the portions of $f$ obtained by integrating over $S_1$ and $S_2$ respectively. Let $N$ be the maximum of $|\mu - \mu_A|$ on $S_1$, then

$$|f_1| \leq N\iint\limits_{S_1}\frac{|\cos\psi|}{r^2}\,dS \leq N\iint\limits_S\frac{|\cos\psi|}{r^2}\,dS.$$

The integral here is bounded; for let $m$ be the maximum number of times $S$ is cut by any straight line, then on account of the geometrical meaning of the integrand, we have uniformly

$$\iint\limits_S\frac{|\cos\psi|}{r^2}\,dS < 4\pi m,$$

so that $|f_1| < 4\pi mN$. But on account of the continuity of $\mu$, $N \to 0$ as $\delta \to 0$. Hence $|f_1|$ may be made as small as we please by taking $\delta$ sufficiently small, and this uniformly for all positions of $P$. But $f_2$ is a continuous function in the neighbourhood of $A$ since the surface $S_2$ does not contain $A$. Hence it is evident that $f$ is continuous at $A$.

## Art. 6. Discontinuity of Normal Derivative at a Surface Distribution

Let a distribution of density $\sigma$ (continuous) on a surface $S$ of continuous curvature produce the potential

$$U = \iint_S \frac{\sigma}{r}\, dS. \tag{19}$$

We choose a direction for the normal $\boldsymbol{n}_A$ at an arbitrary point $A$ of the surface (not a boundary point if the surface is open), and suppose that the field-point $P$ moves along this normal. As long as $P$ is not on $S$, we can differentiate under the integral sign, and obtain

$$\frac{\partial U}{\partial n_A} = \iint_S \sigma \frac{\partial\left(\frac{1}{r}\right)}{\partial n_A}\, dS. \tag{20}$$

We will now investigate the limits of $\dfrac{\partial U}{\partial n_A}$ when $P$ approaches the surface from the positive or negative side, and designate these limits by $\dfrac{\partial U_+}{\partial n_A}$ and $\dfrac{\partial U_-}{\partial n_A}$. We will show that: *the normal derivative has a jump*

$$\frac{\partial U_+}{\partial n_A} - \frac{\partial U_-}{\partial n_A} = -\,4\pi\sigma_A \tag{21}$$

*when $P$ passes through the surface at $A$ in the direction of $\boldsymbol{n}_A$.*

We have

$$\frac{\partial r}{\partial n_A} = \frac{\partial r}{\partial x}\frac{\partial x}{\partial n_A} + \ldots = \frac{x-\xi}{r}\cos(n_A, x) + \ldots$$
$$= \cos(r, x)\cos(n_A, x) + \ldots = \cos(n_A, r)$$
$$(r = (x - \xi, \ldots, \ldots)),$$
$$\frac{\partial\left(\frac{1}{r}\right)}{\partial n_A} = -\frac{1}{r^2}\cos(n_A, r),$$

so that

$$\frac{\partial U}{\partial n_A} = -\iint_S \frac{\sigma}{r^2}\cos(n_A, r)\,dS. \tag{20*}$$

Introducing the normal $\boldsymbol{n}_Q$ at the integration point $Q$, we can write

$$\frac{\partial U}{\partial n_A} = -\iint_S \sigma\frac{\cos(n_Q, r)}{r^2}\,dS + \iint_S \sigma\frac{\cos(n_Q, r) - \cos(n_A, r)}{r^2}\,dS \tag{22}$$
$$= U_1 + U_2.$$

Here the first integral $U_1$ is the potential of a double layer of moment $-\sigma$ and hence from the preceding article we have

$$U_{1+} = E - 2\pi\sigma_A, \quad U_{1-} = E + 2\pi\sigma_A, \tag{23}$$

where

$$E = -\iint_S \sigma\frac{\cos(r_{QA}, n_Q)}{r_{QA}^2}\,dS.$$

We shall complete the proof of the theorem by showing that $U_2$ is continuous as $P$ passes through $S$ at $A$.

Cut out a portion $S_1$ of $S$ by a small sphere of radius $\delta$ about $A$, and let $S_2$ be the remainder of $S$. Then

$$U_2 = \iint_{S_1+S_2} \sigma\frac{\cos(n_Q, r) - \cos(n_A, r)}{r^2}\,dS = F_1 + F_2.$$

Let $N$ be the maximum of $|\sigma|$ on $S$, then

$$|F_1| \leq N \iint_{S_1} \frac{|\cos(n_Q, r_{QP}) - \cos(n_A, r_{QP})|}{r_{QP}^2} dS$$

$$\leq N \iint_{S_1} \frac{|\cos \alpha - \cos \beta|}{r^2} dS$$

$$\leq N \iint_{S_1} \frac{2\left|\sin \dfrac{\alpha - \beta}{2}\right| \left|\sin \dfrac{\alpha + \beta}{2}\right|}{r^2} dS$$

$$\leq 2N \iint_{S_1} \frac{\sin \dfrac{|\alpha - \beta|}{2}}{r^2} dS.$$

By drawing through $Q$ a line parallel to $\boldsymbol{n}_A$ we have at $Q$ a vertex at which three faces meet with the face angles $\alpha$, $\beta$, and $\theta$, where $\theta$ is the angle between $\boldsymbol{n}_A$ and $\boldsymbol{n}_Q$. Hence $\theta \geqq |\alpha - \beta|$, so that

$$|F_1| \leq 2N \iint_{S_1} \frac{\sin(\theta/2)}{r^2} dS \leq a_1 \iint_{S_1} \frac{\sin \theta}{r^2} dS$$

since $\theta$ is a small angle when $S_1$ is small.[2] To simplify this, assume that $A$ is the origin of coordinates with the $z$-axis in the direction $\boldsymbol{n}$, then for a small region such as $S_1$ the equation of the surface may be written in the form $\zeta = \zeta(\xi, \eta)$. (On account of the continuous curvature of $S$ the function $\zeta$ has continuous derivatives of first and second order.) Then the third direction cosine of $\boldsymbol{n}_Q$ is

[2] $a_1$ is a positive constant, as, later, are , $a_6$.

$$\cos\theta = \frac{1}{\sqrt{\left(\frac{\partial\zeta}{\partial\xi}\right)^2 + \left(\frac{\partial\zeta}{\partial\eta}\right)^2 + 1}},$$

so that

$$\sin\theta \leq \sqrt{\left(\frac{\partial\zeta}{\partial\xi}\right)^2 + \left(\frac{\partial\zeta}{\partial\eta}\right)^2}.$$

Since at the origin $A$ we have evidently $\frac{\partial\zeta}{\partial\xi} = \frac{\partial\zeta}{\partial\eta} = 0$, we get by Taylor's formula

$$\frac{\partial\zeta}{\partial\xi} = \xi\frac{\partial^2\zeta(\alpha\xi,\ \alpha\eta)}{\partial\xi^2} + \eta\frac{\partial^2\zeta(\alpha\xi,\ \alpha\eta)}{\partial\xi\partial\eta},\ 0 < \alpha < 1,$$

so that

$$\left|\frac{\partial\zeta}{\partial\xi}\right| \leq a_2\,(|\xi| + |\eta|) < 2a_2r, \text{ and similarly } \left|\frac{\partial\zeta}{\partial\eta}\right| < 2a_2r;$$

because the second derivatives, being continuous, must be bounded in the closed region $S_1$, and evidently

$$|\xi| < r = \sqrt{\xi^2 + \eta^2 + (z - \zeta)^2},\ \text{etc.}$$

Hence $\sin\theta < 3a_2r$, so that

$$|F_1| \leq a_1 \iint\limits_{S_1} \frac{3a_2r}{r^2}\,dS$$

$$\leq a_3 \iint\limits_{S_1} \frac{dS}{r};$$

and from (4), this integral may be made as small as we please by making $\delta$ sufficiently small. But $F_2$ is continuous, so that the change in $F_2$ may be likewise made as small as we please by taking $P$ sufficiently close to $A$. Hence $U_2$ is continuous, completing the proof.

The equations (21) and (23) can be proved under lighter hypotheses on the surface $S$. Using the same axis system as above, introduce cylindrical coordinates $\xi = l\cos\phi$, $\eta = l\sin\phi$, $\zeta$

and assume that the surface in the neighbourhood of the origin $A$ can be expressed in the form

$$\zeta = l^{1+\beta} g(l, \phi), \quad \beta > 0,$$

where $g$ has bounded first partial derivatives. (This hypothesis was fulfilled above with $\beta = 1$.) Now

$$\frac{\partial \zeta}{\partial \xi} = \frac{\xi}{l}\frac{\partial \zeta}{\partial l} - \frac{\eta}{l^2}\frac{\partial \zeta}{\partial \phi}, \quad \frac{\partial \zeta}{\partial \eta} = \frac{\eta}{l}\frac{\partial \zeta}{\partial l} + \frac{\xi}{l^2}\frac{\partial \zeta}{\partial \phi};$$

$$\frac{\partial \zeta}{\partial l} = (1+\beta) l^\beta g + l^{1+\beta}\frac{\partial g}{\partial l}, \quad \frac{\partial \zeta}{\partial \phi} = l^{1+\beta}\frac{\partial g}{\partial \phi}.$$

From this it follows that $\left|\frac{\partial \zeta}{\partial \xi}\right| < a_4 l^\beta$, $\left|\frac{\partial \zeta}{\partial \eta}\right| < a_4 l^\beta$; and hence $\sin\theta < a_5 l^\beta < a_5 r^\beta$, since obviously $l \leq r$.

We thus obtain
$$|F_1| < a_6 \iint_{S_1} \frac{dS}{r^{2-\beta}}$$
which may be made arbitrarily small with $\delta$, from (4*).

*Remark*: The value at $A$ of $U_2$ is

$$U_2(A) = \iint_S \sigma \frac{\cos(n_Q, r_{QA}) - \cos(n_A, r_{QA})}{r_{QA}^2} dS.$$

Taking into account the continuity of $U_2$ at $A$ we get from (22) and (23)

$$\frac{\partial U_+}{\partial n_A} = U_{1+} + U_2(A) = -2\pi\sigma_A + E + U_2(A),$$

$$\frac{\partial U_-}{\partial n_A} = U_{1-} + U_2(A) = 2\pi\sigma_A + E + U_2(A).$$

Since obviously

$$E + U_2(A) = -\iint_S \frac{\cos(n_A, r_{QA})}{r_{QA}^2} dS$$

we obtain the two equations

$$(21^*)\qquad \begin{aligned} \frac{\partial U_+}{\partial n_A} &= -2\pi\sigma_A - \iint_S \frac{\cos(n_A, r_{QA})}{r_{QA}^2}\, dS \\ \frac{\partial U_-}{\partial n_A} &= 2\pi\sigma_A - \iint_S \frac{\cos(n_A, r_{QA})}{r_{QA}^2}\, dS. \end{aligned}$$

From these follow not only equation (21) of our theorem but

$$(21^{**})\qquad \tfrac{1}{2}\left(\frac{\partial U_+}{\partial n_A} + \frac{\partial U_-}{\partial n_A}\right) = -\iint_S \frac{\cos(n_A, r_{QA})}{r_{QA}^2}\, dS.$$

## Art. 7. Normal Derivative of Potential of a Double Layer

We will next investigate the behaviour of the normal derivative of the potential

$$U = \iint_S \mu \frac{\partial\left(\frac{1}{r}\right)}{\partial n}\, dS = \iint_S \mu \frac{\cos(n, r)}{r^2}\, dS$$

of a double layer of continuous moment $\mu$ at points in the neighbourhood of $S$. Assume at first that $S$ is closed, has continuous curvature, and that $\boldsymbol{n}$ is directed outward. Let $P_1$ and $P_2$ be points on the negative and positive sides of $S$, located on the normal $\boldsymbol{n}_A$ at a particular point $A$ of $S$ at *equal distances* $\delta$ from $A$. Then we will first show that

$$(24)\qquad \left(\frac{\partial U}{\partial n}\right)_{P_2} - \left(\frac{\partial U}{\partial n}\right)_{P_1} \to 0 \text{ as } \delta \to 0.$$

Of course it does not follow from this that the separate limits of $\left(\frac{\partial U}{\partial n}\right)_{P_1}$ and $\left(\frac{\partial U}{\partial n}\right)_{P_2}$ exist. However, we will prove later that this is true under more restrictive hypotheses on $\mu$. Of course it does follow from (24) that *if one of the separate limits*

*of* $\left(\dfrac{\partial U}{\partial n}\right)_{P_i}$ *exists, then the other must also exist and must have the same value.*

For the proof we again assume that the point $A$ and the normal at $A$ are the origin and direction of the $z$-axis respectively, so that the $(x, y)$-plane is tangent to $S$ at $A$. The potential at points P:(0, 0, $z$) on the normal then becomes a function of $z$, say

$$U(0,\, 0,\, z) = F(z). \tag{25}$$

Then we wish to prove that

$$F'(z) - F'(-z) \to 0 \text{ as } z \to 0. \tag{24*}$$

We can assume without loss of generality that $\mu$ vanishes at $A$, since

$$U = \iint_S \mu_A \frac{\partial\left(\frac{1}{r}\right)}{\partial n}\, dS + \iint_S (\mu - \mu_A) \frac{\partial\left(\frac{1}{r}\right)}{\partial n}\, dS;$$

the first integral here has the value $-4\pi\mu_A$ at all inside points, and the value 0 at all outside points, so that its normal derivatives are both zero and hence satisfy (24). The second integral is the potential of a double layer which has the moment 0 at $A$.

The equation of the surface in the neighbourhood of $A$ can be written

$$\zeta = \zeta(\xi, \eta), \tag{26}$$

where this function has continuous second partial derivatives. Then at the origin we have $\xi = \eta = \zeta\ (0, 0) = \dfrac{\partial \zeta}{\partial \xi} = \dfrac{\partial \zeta}{\partial \eta} = 0$ so that

$$\zeta = \xi^2 \frac{\partial^2 \zeta\ (\alpha\xi, \alpha\eta)}{\partial \xi^2} + 2\xi\eta \frac{\partial^2 \zeta\ (\alpha\xi, \alpha\eta)}{\partial \xi \partial \eta} + \eta^2 \frac{\partial^2 \zeta\ (\alpha\xi, \alpha\eta)}{\partial \eta^2},\ 0 < \alpha < 1.$$

Introducing cylindrical coordinates

$$\xi = l \cos \phi, \ \eta = l \sin \phi, \ \zeta = \zeta,$$

it is evident that for a sufficiently small $l_0$, there is an $M$ such that

$$(27) \qquad |\zeta| < l^2 M, \left|\frac{\partial \zeta}{\partial l}\right| < lM, \text{ for } 0 < l \leq l_0.$$

The value of $l_0$ will be fixed later. Let $S_1$ be the part of $S$ near $A$ for which $l \leq l_0$ and let $S_2$ be the remainder of $S$, and let

$$(28) \quad U = \iint_{S_1} \mu \frac{\cos(r, n)}{r^2} dS + \iint_{S_2} \mu \frac{\cos(r, n)}{r^2} dS = U_1 + U_2.$$

Since for the normal at $Q$ we have $n_1 : n_2 : n_3 = \dfrac{\partial \zeta}{\partial \xi} : \dfrac{\partial \zeta}{\partial \eta} : -1$,

then $\cos(r, n)\, dS = \left[ n_1 \cdot \dfrac{(x-\xi)}{r} + n_2 \cdot \dfrac{(y-\eta)}{r} + n_3 \cdot \dfrac{(z-\zeta)}{r} \right] dS$

$$= \frac{(\xi - x)\dfrac{\partial \zeta}{\partial \xi} + (\eta - y)\dfrac{\partial \zeta}{\partial \eta} + z - \zeta}{r\sqrt{1 + \left(\dfrac{\partial \zeta}{\partial \xi}\right)^2 + \left(\dfrac{\partial \zeta}{\partial \eta}\right)^2}} dS,$$

then

$$U_1 = \iint \mu \frac{(\xi - x)\dfrac{\partial \zeta}{\partial \xi} + (\eta - y)\dfrac{\partial \zeta}{\partial \eta} + z - \zeta}{r^3} d\xi\, d\eta,$$

where the integration is over the projection of $S_1$ on the $(x, y)$-plane. Hence, for $P$ on the $z$-axis this becomes

$$(29) \qquad F_1(z) = \iint \mu \frac{\xi \dfrac{\partial \zeta}{\partial \xi} + \eta \dfrac{\partial \zeta}{\partial \eta} + z - \zeta}{r^3} d\xi\, d\eta = U_1(0, 0, z),$$

or

$$(29^*) \qquad F_1(z) = \iint \mu \frac{l \dfrac{\partial \zeta}{\partial l} + z - \zeta}{[l^2 + (z - \zeta)^2]^{3/2}} l\, dl\, d\phi.$$

For compactness we let $g = \dfrac{l\dfrac{\partial\zeta}{\partial l} + z - \zeta}{r^3}$; then

$$(29^{**}) \qquad F_1(z) = \int_0^{l_0} l\,dl \int_0^{2\pi} \mu g\,d\phi$$

and hence

$$(30) \qquad F_1'(z) = \int_0^{l_0} l\,dl \int_0^{2\pi} \mu \frac{\partial g}{\partial z}\,d\phi.$$

Let $r_0 = \sqrt{l^2 + z^2}$, $\lambda = \dfrac{r^2}{r_0{}^2} - 1$, and recall that

$$r = \sqrt{l^2 + (z - \zeta)^2}\,.$$

When the point $Q{:}(\xi, \eta, \zeta)$ approaches the origin, we have $\zeta \to 0$, $r \to |z|$, $r_0 \to |z|$ so that $r \to r_0$, and $\lambda \to 0$. Finally define $N$ by the equation

$$\frac{r_0{}^5}{r^5} = 1 - N\lambda.$$

Since $\dfrac{r^2}{r_0{}^2} = 1 + \lambda$, we have

$$\frac{r_0^5}{r^5} = (1 + \lambda)^{-5/2} = 1 - \frac{5}{2}\lambda + \cdots$$

for $|\lambda| < 1$, from which it follows that $N \to \dfrac{5}{2}$; hence if $l_0$ and therefore $\lambda$ is made small enough, we will have $2 < N < 3$.

We now investigate the integral

$$(31) \quad F_1'(z) - \int_0^{l_0}\int_0^{2\pi} l\frac{l^2 - 2z^2}{r_0^5}\mu\,dl\,d\phi = \int_0^{l_0}\int_0^{2\pi} l\left(\frac{\partial g}{\partial z} - \frac{l^2 - 2z^2}{r_0^5}\right)\mu\,dl\,d\phi,$$

and need therefore an inequality for

$$l\left(\frac{\partial g}{\partial z} - \frac{l^2 - 2z^2}{r_0^5}\right) = l\left[\frac{l^2 - 2(z-\zeta)^2}{r^5} - \frac{l^2 - 2z^2}{r_0^5}\right] - \frac{3l^2(z-\zeta)}{r^5}\frac{\partial \zeta}{\partial l}.$$

Now
$$\left|\frac{3l^2\zeta}{r^5}\frac{\partial \zeta}{\partial l}\right| \leq \frac{3l^2|\zeta|}{l^5}\left|\frac{\partial \zeta}{\partial l}\right| < 3M^2 \text{ from (27);}$$

$$\left|\frac{3l^2 z}{r^5}\frac{\partial \zeta}{\partial l}\right| < \frac{3|z|M}{r^2} < \frac{4|z|M}{r_0^2}$$

since, for $\dfrac{r}{r_0} \to 1$, we can choose $l_0$ small enough so that

$$\frac{r_0^2}{r^2} < \frac{4}{3} \text{ for } 0 < l < l_0.$$

Moreover,

$$\begin{aligned} &l\left[\frac{l^2 - 2(z-\zeta)^2}{r^5} - \frac{l^2 - 2z^2}{r_0^5}\right] \\ &= \frac{l}{r_0^5}[(l^2 - 2(z-\zeta)^2)(1 - N\lambda) - l^2 + 2z^2] \\ &= \frac{l}{r_0^5}[-N\lambda(l^2 - 2(z-\zeta)^2) - 2\zeta^2 + 4z\zeta] \\ &= -\frac{l\lambda}{r_0^5}[N(l^2 - 2(z-\zeta)^2) + 2r_0^2], \end{aligned}$$

since $-2\zeta^2 + 4z\zeta = -2(r^2 - r_0^2) = -2\lambda r_0^2$. Now

$$\begin{aligned} N(l^2 - 2(z-\zeta)^2) + 2r_0^2 &< 2r_0^2 + 2N(l^2 + (z-\zeta)^2) \\ &= 2r_0^2 + 2Nr^2 < 2r_0^2 + 6r^2 < 9r_0^2, \end{aligned}$$

for, since $\dfrac{r}{r_0} \to 1$, we can assume that $\dfrac{r^2}{r_0^2} < \dfrac{7}{6}$ for $l_0$ sufficiently small. From this it follows that

$$\begin{aligned} \left|l\left[\frac{l^2 - 2(z-\zeta)^2}{r^5} - \frac{l^2 - 2z^2}{r_0^5}\right]\right| &< \frac{9l|\lambda|r_0^2}{r_0^5} \\ &< \frac{9|\lambda|}{r_0^2} < 9\left(M^2 + 2M\frac{|z|}{r_0^2}\right), \end{aligned}$$

if we note that

$$|\lambda| = \frac{|\zeta^2 - 2z\zeta|}{r_0^2} \leq \frac{\zeta^2 + 2|z\zeta|}{l^2} < M^2l^2 + 2M|z|$$
$$< M^2r_0^2 + 2M|z| .$$

From the above results, we find

$$\left| l\left(\frac{\partial g}{\partial z} - \frac{l^2 - 2z^2}{r_0^5}\right)\right| < 3M^2 + \frac{4|z|M}{r_0^2} + 9\left(M^2 + 2M\frac{|z|}{r_0^2}\right)$$
$$< 12M^2 + 24M\frac{|z|}{r_0^2}.$$

Hence

$$\left| F_1'(z) - \int_0^{l_0}\int_0^{2\pi} l\frac{l^2 - 2z^2}{r_0^5}\mu dld\phi \right| <$$
$$\int_0^{l_0}\int_0^{2\pi}\left(12M^2 + 24M\frac{|z|}{r_0^2}\right)|\mu|dld\phi$$
$$< \mu_m\left[12M^2\int_0^{l_0}\int_0^{2\pi} dld\phi + 24M\int_0^{l_0}\int_0^{2\pi}\frac{|z|}{l^2 + z^2}dld\phi\right]$$

where $\mu_m$ is the maximum value of $|\mu|$ in $S_1$. But

$$\int_0^{l_0}\frac{|z|}{l^2 + z^2}dl < \int_0^{\infty}\frac{|z|}{l^2 + z^2}dl = \int_0^{\infty}\frac{dt}{1 + t^2} = \text{arc tan}\,\infty = \frac{\pi}{2},$$

since $|z| \neq 0$. We get finally

$$\left| F_1'(z) - \int_0^{l_0}\int_0^{2\pi} l\frac{l^2 - 2z^2}{r_0^5}\mu dld\phi \right| < \frac{c}{2}\mu_m, \tag{32}$$

where $c = 48M^2\pi l_0 + 48M\pi^2$. Exactly the same result is reached if we replace $z$ by $-z$, since $l\dfrac{l^2 - 2z^2}{r_0^5}$ is an even function of $z$; hence

$$\left| F_1'(-z) - \int_0^{l_0}\int_0^{2\pi} l\frac{l^2 - 2z^2}{r_0^5}\mu dld\phi \right| < \frac{c}{2}\mu_m. \tag{32*}$$

By combining (32) and (32*), we get

$$|F_1'(z) - F_1'(-z)| < c\mu_m. \tag{33}$$

Now it is easy to prove (24) or (24*). Given an arbitrary positive $\epsilon$, we first choose $l_0$ small enough so that the maximum $\mu_m$ of $|\mu|$ on $S_1$ satisfies the inequality $c\mu_m < \frac{\epsilon}{2}$, which is possible because $\mu$ is continuous and vanishes at $A$, and $c$ is bounded no matter how small $l$ may be. It follows that

$$|F_1'(z) - F_1'(-z)| < \frac{\epsilon}{2}, \text{ for } z \neq 0.$$

But for $F_2(z)$, $A$ is not on the integration surface, so that $F_2(z)$ is continuous and we can find a $\delta$ small enough so that for $0 \leq |z| \leq \delta$,

$$|F_2'(z) - F_2'(-z)| < \frac{\epsilon}{2}.$$

We finally obtain for $0 < |z| \leq \delta$

$$|F'(z) - F'(-z)| < \epsilon, \tag{34}$$

the desired result.

In case $S$ is an open surface instead of closed, we proceed as before to extend it to a closed surface with the definition of $\mu$ extended over it in a continuous manner; it is then seen that the equation (24) is valid for open surfaces as well as for closed ones.

We now assume that *the moment* $\mu$ on $S$ is not merely continuous, but that it *has continuous first and second derivatives. Then the normal derivative of the potential of a double layer*

$$U = \iint_S \mu \frac{\partial\left(\frac{1}{r}\right)}{\partial n} dS \tag{35}$$

*approaches a limit as $P$ approaches the surface from either side at any point $A$ not on an edge of the surface $S$.* From the preceding result these two limits must then be equal.

To prove this, suppose that $P$ lies on the positive side on the normal at $A$, and surround $A$ by a small sphere which contains $P$ in its interior; designate that portion of the volume of the sphere which lies on the positive side of $S$ by $V$, and let $S'$ be the total surface of the volume $V$. $S'$ consists of a part $S_1$ of $S$ in the neighbourhood of $A$, and a part $S$ of the surface of the sphere. Suppose that the function $\mu$ which is defined on $S$ is extended into the space $V$, so that this extended function has continuous second derivatives. In other words, we use a function defined in $V$ with continuous second derivatives, which becomes the given function $\mu$ on $S_1$. For example, the function could be extended in such a way that it remains constant along each normal, or on parallels to the normal at $A$. By applying Green's formula (34) of Chapter III to the function $\mu$ and the volume $V$, we get

$$\text{(36)}\quad \iint\limits_{S'} \mu \frac{\partial\left(\frac{1}{r}\right)}{\partial n} dS = \iint\limits_{S_1} \mu \frac{\partial\left(\frac{1}{r}\right)}{\partial n} dS + \iint\limits_{\bar{S}} \mu \frac{\partial\left(\frac{1}{r}\right)}{\partial n} dS$$
$$\iint\limits_{S'} \frac{1}{r} \frac{\partial \mu}{\partial n} dS - \iiint\limits_{V} \frac{1}{r} \nabla^2 \mu \, dV - 4\pi\mu\,(P)$$

Being the potential of a continuous space distribution, $\iiint\limits_{V} \frac{1}{r} \nabla^2 \mu \, dV$ has continuous derivatives everywhere, and so does the function $-4\pi\mu(P)$ and the function $\iint\limits_{\bar{S}} \mu \frac{\partial\left(\frac{1}{r}\right)}{\partial n} dS$ since $A$ is not on the surface $S$. Also the potential of a surface

distribution $\iint\limits_{S'} \frac{1}{r}\frac{\partial \mu}{\partial n}\, dS$ has from Art. 6 the property that the normal derivative has a limit on approaching the surface. Accordingly this follows for the one remaining term

$$\iint\limits_{S_1} \mu \frac{\partial\left(\frac{1}{r}\right)}{\partial n}\, dS,$$

and hence for

$$U = \iint\limits_{S} \mu \frac{\partial\left(\frac{1}{r}\right)}{\partial n}\, dS, \tag{37}$$

as we wanted to prove. Likewise the normal derivative of $U$ has a limit when $P$ approaches $A$ from the negative side; and these two limits are the same.

## Art. 8. Analogous Theorems for Logarithmic Potential

The following theorems for logarithmic potential can be proved by the methods used in the last paragraphs.

*The potential*

$$U = \iint\limits_{T} \sigma \log \frac{1}{r}\, dS, \tag{38}$$

*where $T$ is a region bounded by a closed Jordan curve*[3] *and $\sigma$ is bounded and integrable in $T$, is finite and continuous in the entire plane except at infinity. This is true also for the first derivatives, which are given by*

$$\frac{\partial U}{\partial x} = \iint\limits_{T} \sigma \frac{\partial\left(\log \frac{1}{r}\right)}{\partial x}\, dS, \quad \frac{\partial U}{\partial y} = \iint\limits_{T} \sigma \frac{\partial\left(\log \frac{1}{r}\right)}{\partial y}\, dS. \tag{39}$$

[3]Or by several Jordan curves. The region may of course be multiply-connected.

*If the boundary curve C of T has a continuously turning tangent and the density $\sigma$ has continuous first derivatives in T, the second derivatives of U are continuous both inside and outside T, but not on its boundary. They can be found from*

$$(40)\qquad \begin{aligned} \frac{\partial^2 U}{\partial x^2} &= \iint_T \frac{\partial \sigma}{\partial \xi} \frac{\partial\left(\log \frac{1}{r}\right)}{\partial x} dS + \oint_C \sigma \cos(n, x) \frac{\partial\left(\log \frac{1}{r}\right)}{\partial x} ds, \\ \frac{\partial^2 U}{\partial x \partial y} &= \iint_T \frac{\partial \sigma}{\partial \xi} \frac{\partial\left(\log \frac{1}{r}\right)}{\partial y} dS + \oint_C \sigma \cos(n, x) \frac{\partial\left(\log \frac{1}{r}\right)}{\partial y} ds, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

*In the region T the potential U satisfies Poisson's equation*

$$(41)\qquad \nabla^2 U = -2\pi\sigma.$$

If the hypotheses on $\sigma$ are not satisfied in the entire region $T$, these properties still hold in the sub-regions where they are satisfied.

*The potential of a simple distribution on a curve*

$$(42)\qquad U = \int_C \gamma \log \frac{1}{r} ds,$$

*if the curve has a continuously turning tangent and $\gamma$ is bounded and integrable, is continuous for all finite points of the plane including passage through C. If C has continuous curvature and $\gamma$ is continuous, then the normal derivatives of U approach limits when P approaches A on C from either the positive or negative side, which satisfy the equations*

$$(43)\qquad \begin{aligned} \frac{1}{2}\left(\frac{\partial U_+}{\partial n_A} - \frac{\partial U_-}{\partial n_A}\right) &= -\pi \gamma_A, \\ \frac{1}{2}\left(\frac{\partial U_+}{\partial n_A} + \frac{\partial U_-}{\partial n_A}\right) &= \int_C \gamma \frac{\partial \log \dfrac{1}{r_{As}}}{\partial n_A} ds. \end{aligned}$$

*If the curve $C$ is not closed, it is assumed that $A$ is not an end-point of $C$.*

If the hypotheses of continuous curvature of $C$ and continuity of $\gamma$ are only fulfilled on portions of $C$, the equations (43) hold on these portions, end-points being excluded. This is easily proved in the usual manner by regarding $U$ as the sum of two potentials. Instead of continuous curvature, it is sufficient to assume that $C$ can be expressed in the form

$$y = x^{1+a}g(x)$$

where $a > 0$ and $g(x)$ has a finite derivative, when the point $A$ and the tangent are taken as the origin and the $x$-axis respectively.

*The potential of a double distribution on a line $C$,*

$$U = \int_C \mu \frac{\partial\left(\log \frac{1}{r}\right)}{\partial n} ds = \int_C \mu \frac{\cos(n, r)}{r} ds, \tag{44}$$

*if $C$ has a continuously turning tangent and $\mu$ is continuous, has limits on approaching $C$ from either side (except at its end-points) which are in general different from each other and satisfy*

$$\tfrac{1}{2}(U_+ - U_-) = \pi\mu_A, \quad \tfrac{1}{2}(U_+ + U_-) = U_A. \tag{45}$$

If we permit $C$ to have a finite number of corners and $A$ is at a corner, then

$$\tfrac{1}{2}(U_+ - U_-) = \pi\mu_A, \quad \tfrac{1}{2}(U_+ + U_-) = (\pi - \beta)\mu_A + U_A, \tag{45*}$$

where $\beta$ is the angle between tangents at $A$.

If $C$ has continuous curvature, we can prove that the normal derivatives of (44) satisfy

$$\lim_{\delta \to 0} \left\{\left(\frac{\partial U}{\partial n}\right)_{P_2} - \left(\frac{\partial U}{\partial n}\right)_{P_1}\right\} = 0, \tag{46}$$

where $P_1$ and $P_2$ are at the same distance $\delta$ from $C$ on opposite

sides. *Finally, by assuming that $\mu$ has a continuous derivative, we can prove that the separate limits* $\left(\frac{\partial U}{\partial n}\right)_{P_1}$ *exist and are equal.*

**Art. 9. Dirichlet's Characteristic Properties of Potential**

We are now able to derive the characteristic properties of the Newtonian potential

$$U = \iiint \frac{\tau dV}{r},$$

which were first given by Dirichlet. These conditions are characteristic, in that they are necessary and sufficient conditions that any function which satisfies them be identical with a potential $U$ of a mass distribution. We assume that the density $\tau$ is piece-wise continuous with piece-wise continuous first derivatives in the whole of space and vanishes identically outside a sufficiently large sphere about the origin, so that the above integral may be supposed to be extended over all space.

*The characteristic properties are the following:*

I. *The function $U$ is continuous over all space with continuous first derivatives.*

II. *The second derivatives are continuous everywhere except on surfaces of discontinuity of* $\tau$, $\frac{\partial \tau}{\partial x}$, $\frac{\partial \tau}{\partial y}$, $\frac{\partial \tau}{\partial z}$. *They satisfy* $\nabla^2 U = -4\pi\tau$.

III. *For* $R = \sqrt{x^2+y^2+z^2} \to \infty$, $U \to 0$ *and* $R^2|D_1 U|$ *is bounded.*

Instead of the boundedness of $R^2|D_1 U|$, it is possible to use the condition that $U$ behaves at infinity like the potential of a mass, more exactly, that

$$U = \frac{M}{R} + u \qquad \left(M = \iiint \tau dV\right)$$

where $u \to 0$ and $R^3|D_1 u|$ is bounded as $R \to \infty$.

That the three conditions are necessary, or that the potential $\iiint \frac{\tau}{r} dV$ satisfies them, has been proved in Arts. 2, 3, and in Chapter II, Art. 6.

That they are sufficient, i.e. that a function $U$ which satisfies them is identical with a potential $\iiint \frac{\tau}{r} dV$, may be shown in the following manner. On account of the properties assumed in I and II, we have from (34), Chapter III:

$$4\pi U = \iint_S \left( \frac{1}{r} \frac{\partial U}{\partial n} - U \frac{\partial \left( \frac{1}{r} \right)}{\partial n} \right) dS - \iiint_V \frac{\nabla^2 U}{r} dV.$$

Here $V$ is any region containing $P$ in its interior. We choose for $V$ a large sphere about the origin, and let its radius approach infinity. The surface integral vanishes in the limit on account of the conditions III.

Hence we obtain in the limit

$$4\pi U = -\iiint \frac{\nabla^2 U}{r} dV$$

where the integral is over all space, and since $\nabla^2 U = -4\pi\tau$ from II, we have

$$U = \iiint \frac{\tau}{r} dV,$$

which is the desired result.

Dirichlet used these characteristic properties on the famous problem of the determination of the potential of a homogeneous ellipsoid (see next Art.).

For the logarithmic potential

$$U = \iint \sigma \log \left( \frac{1}{r} \right) dS,$$

in which $\sigma$ and its first derivatives are piece-wise continuous in the entire plane and vanish outside a sufficiently large circle

about the origin, the following properties are characteristic:
I. The function $U$ and its first derivatives are continuous everywhere in the finite plane.
II. The second derivatives are continuous everywhere except on lines of discontinuity of $\sigma, \frac{\partial \sigma}{\partial x}, \frac{\partial \sigma}{\partial y}$. They satisfy $\nabla^2 U = -2\pi\sigma$.
III. For $R = \sqrt{x^2+y^2} \to \infty$, $\left(U + R \log \frac{1}{R} \frac{\partial U}{\partial R}\right) \to 0$ and $R|D_1 U|$ is bounded.

Instead of the last condition we can use the condition that $U$ behave at infinity like the potential of a mass, more exactly that $U$ is of the form

$$U = M \log \frac{1}{R} + u \quad \left(M = \iint \sigma dS\right)$$

where $u \to 0$ and $R^2|D_1 u|$ is bounded.

The necessity of these conditions follows from Art. 8 for I and II, and from Art. 6, Chapter II, for III. In particular, it follows readily from the representation $U = M \log \frac{1}{R} + u$ with $u \to 0$ and $R^2|D_1 u|$ bounded that $U + R \log \frac{1}{R} \frac{\partial U}{\partial R} \to 0$, since $U + R \log \frac{1}{R} \frac{\partial U}{\partial R} = u + R \log \frac{1}{R} \frac{\partial u}{\partial R}$ and $u \to 0$,

$$R \log R \frac{\partial u}{\partial R} \to 0.$$

That the above conditions are also sufficient can be proved as in the case of Newtonian potential above (the reader should carry this out). We will later (Chapter VIII, Art. 5) prove that the existence of lim $u$ implies the boundedness of $R^2|D_1 u|$; hence the last condition may be written simply

III*. For , $U = M \log \frac{1}{R} + u$ where $u \to 0$.

**Art. 10. Applications**

1. In Chapter II, Art. 9, we found the potential of a mass distribution bounded by two concentric spheres, with density $\rho$ a function of the distance $R$ from their centres, at points outside the outer sphere and at points inside the inner one. We can now find the potential at the points inside the mass.

We will first find the value of the constant $b$ found there, as the potential inside the inner sphere. This can readily be found by integration, as it is the value of the potential at the centre, which we will take as origin. If $l_1$ and $l_2$ are the radii of the spheres ($l_1 < l_2$),

$$b = U_0 = \iiint \frac{\rho}{R} dV = 4\pi \int_{l_1}^{l_2} R\rho(R) dR.$$

If the mass is homogeneous (or $\rho$ constant), this becomes

$$b = 2\pi\rho(l_2^2 - l_1^2).$$

At a point of the mass, the potential satisfies Poisson's equation, and since $U$ only depends on $R$, this is

$$\frac{d^2U}{dR^2} + \frac{2}{R}\frac{dU}{dR} = -4\pi\rho.$$

This is an ordinary linear differential equation of the second order, non-homogeneous. The corresponding homogeneous equation has already been integrated and has the general solution $\frac{A}{R} + B$. By adding this to any particular solution of the non-homogeneous equation, we get its general solution. Hence when $\rho$ is a constant, we find

$$U = -\frac{2\pi\rho R^2}{3} + \frac{A}{R} + B,$$

since $-2\pi\rho R^2/3$ is easily seen to be a particular solution of the non-homogeneous equation.

The integration constants $A$ and $B$ are determined by the continuity properties of the potential. The potential $U(R)$ must be continuous on passing through each boundary sphere, so that

$$-\frac{2\pi\rho l_1^2}{3}+\frac{A}{l_1}+B = 2\pi\rho(l_2^2-l_1^2)$$

$$-\frac{2\pi\rho l_2^2}{3}+\frac{A}{l_2}+B = \frac{4\pi\rho}{3l_2}(l_2^3-l_1^3) = \frac{M}{l_2}\,.$$

These equations have the solutions

$$A = -\frac{4\pi\rho l_1^3}{3},\quad B = 2\pi\rho l_2^2$$

so that if $\rho$ is constant we have the potential function:

$$U = \begin{cases} 2\pi\rho(l_2^2-l_1^2) & \text{for } 0 \leqq R \leqq l_1 \\ -\dfrac{2\pi\rho R^2}{3}-\dfrac{4\pi\rho l_1^3}{3}\dfrac{1}{R}+2\pi\rho l_2^2 & \text{"}\ \ l_1 \leqq R \leqq l_2 \\ \dfrac{4\pi\rho(l_2^3-l_1^3)}{3}\dfrac{1}{R} & \text{"}\ \ l_2 \leqq R. \end{cases}$$

For the special case of a complete sphere of radius $l$ and uniform density, we set $l_1 = 0$, $l_2 = l$ and get

$$U = \begin{cases} -\dfrac{2\pi\rho}{3}R^2+2\pi\rho l^2 & \text{for } 0 \leqq R \leqq l \\ \dfrac{4\pi\rho l^3}{3}\dfrac{1}{R} & \text{"}\ \ l \leqq R. \end{cases}$$

Of course the result can be obtained also by direct integration. Split up the mass distribution into thin shells bounded by two concentric spheres of radii $l$ and $l + dl$. Show that the potential of such a shell is $\dfrac{4\pi\rho(l)l^2dl}{R}$, if $R \geqq l$, and $\dfrac{4\pi\rho(l)l^2}{l}\,dl =$ $4\pi\rho(l)l\,dl$, if $R \leqq l$. Prove that the potential of the given mass is

$$U = 4\pi \int_{l_1}^{l_2} \rho(l) l dl \qquad \text{for } R \leq l_1$$

$$U = \frac{4\pi}{R} \int_{l_1}^{R} \rho(l) l^2 dl + 4\pi \int_{R}^{l_2} \rho(l) l dl \qquad \text{for } l_1 \leq R \leq l_2$$

$$U = \frac{4\pi}{R} \int_{l_1}^{l_2} \rho(l) l^2 dl = \frac{M}{R} \qquad \text{for } l_2 \leq R.$$

Put $\rho$ = const. and find again the former formulas.

2. Mass distribution bounded by two concentric circles attracting by the inverse first power law with density dependent only on the distance from their centre. Find the potential at a point inside the mass.

3. Solid homogeneous ellipsoid. Density $\tau$ = const. Equation of the boundary surface $E$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \qquad a \geq b \geq c. \tag{47}$$

The potential $U$ at any point $P(x, y, z)$ is given by a triple integral in the usual way but we shall represent it by a simple elliptic integral. The equation

$$f(P, u) = \frac{x^2}{a^2 + u} + \frac{y^2}{b^2 + u} + \frac{z^2}{c^2 + u} = 1 \tag{48}$$

represents a set of ellipsoids $E(u)$. The parameter $u$ is supposed to range through all non-negative values. $E$ is obviously identical with $E(0)$. One and only one $E(u)$ passes through any point $P$ outside $E$. Accordingly $u$ with the restriction $u \geq 0$ is defined by (48) as an implicit one-valued function of $P(x, y, z)$.[4] Let $g(u) = \sqrt{(a^2+u)(b^2+u)(c^2+u)}$ and $\tau abc\pi = k$. We state:

$$U(P) = k \int_0^{\infty} \frac{(1 - f(P, v))}{g(v)} dv \qquad \text{for } P \text{ inside } E, \tag{49$_1$}$$

[4]It is one of the 3 ellipsoidal coordinates of $P$. We do not use the others.

$$(49_2) \quad U(P) = k\int_u^\infty \frac{(1 - f(P, v))}{g(v)}\,dv \qquad \text{for } P \text{ outside } E.$$

We sketch Dirichlet's proof that the characteristic conditions (Art. 9) are satisfied.

I. $U$ is obviously continuous over all space, even as P passes through $E$, since $u = 0$ on $E$. For $P$ inside or outside $E$ respectively

$$(50_1) \qquad \frac{\partial U}{\partial x} = -2kx\int_0^\infty \frac{dv}{(a^2+v)g(v)},$$

and
$$\frac{\partial U}{\partial x} = -2kx\int_u^\infty \frac{dv}{(a^2+v)\,g(v)} - k\frac{1 - f(P, u)}{g(u)}\,\frac{\partial u}{\partial x},$$

but $1 - f(P, u) = 0$ since indeed $P$ lies on the ellipsoid $E(u)$. Thus for $P$ outside $E$

$$(50_2) \qquad \frac{\partial U}{\partial x} = -2kx\int_u^\infty \frac{dv}{(a^2+v)\,g(v)}.$$

Therefore $\dfrac{\partial U}{\partial x}$ and likewise $\dfrac{\partial U}{\partial y}$ and $\dfrac{\partial U}{\partial z}$ are continuous everywhere.

II. $$(51_1) \quad \frac{\partial^2 U}{\partial x^2} = -2k\int_0^\infty \frac{dv}{(a^2+v)\,g(v)} \quad \text{for } P \text{ inside } E.$$

$$(51_2) \quad \frac{\partial^2 U}{\partial x^2} = -2k\int_u^\infty \frac{dv}{(a^2+v)\,g(v)} + \frac{2kx}{(a^2+u)g(u)}\,\frac{\partial u}{\partial x}$$ for $P$ outside $E$. Therefore $\dfrac{\partial^2 U}{\partial x^2}$ and likewise $\dfrac{\partial^2 U}{\partial y^2}$ and $\dfrac{\partial^2 U}{\partial z^2}$ are continuous everywhere except at the points of $E$. Moreover, for inner points

$$\Delta U = -2k\int_0^\infty \left(\frac{1}{a^2+v} + \frac{1}{b^2+v} + \frac{1}{c^2+v}\right)\frac{dv}{g(v)}.$$

Prove that

$$(52)\qquad \frac{1}{g(v)}\,\frac{dg}{dv} = \frac{d\log g(v)}{dv} = \frac{1}{2}\left(\frac{1}{a^2+v}+\frac{1}{b^2+v}+\frac{1}{c^2+v}\right)$$

and infer

$$(53_1)\qquad \triangle U = -4k\int_{g(0)}^{\infty}\frac{dg}{g^2} = \frac{-4k}{g(0)} = -4\pi\tau.$$

For $P$ outside $E$

$$\triangle U = -2k\int_u^{\infty}\left(\frac{1}{a^2+v}+\frac{1}{b^2+v}+\frac{1}{c^2+v}\right)\frac{dv}{g(v)}$$

$$+\frac{2k}{g(u)}\left\{\frac{x\dfrac{\partial u}{\partial x}}{a^2+u}+\frac{y\dfrac{\partial u}{\partial y}}{b^2+u}+\frac{z\dfrac{\partial u}{\partial z}}{c^2+u}\right\}.$$

The first term on the right equals $-\dfrac{4k}{g(u)}$ (use again (52)). To reduce the second term find the partial derivatives of $u$ by differentiating (48), for instance

$$\frac{2x}{a^2+u}-\left(\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right)\frac{\partial u}{\partial x} = 0.$$

Determine $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial u}{\partial y}$, $\dfrac{\partial u}{\partial z}$ and prove that the second term equals $\dfrac{4k}{g(u)}$. Finally

$$(53_2)\qquad \triangle U = 0.$$

III. If we put $\sqrt{x^2+y^2+z^2} = R$ then obviously $\sqrt{c^2+u} \leq R \leq \sqrt{a^2+u}$, for the distance of the point $P$ from the origin must lie between the smallest and largest semi-axes. Since $\lim\limits_{\rightarrow\infty}\dfrac{\sqrt{c^2+u}}{\sqrt{a^2+u}} = 1$, we have obviously

$$\lim \frac{\sqrt{c^2+u}}{R} = \lim \frac{\sqrt{a^2+u}}{R} = 1,$$

so that the semi-axes and $R$ are of the same order for large $u$. Now consider the potential $U$ in $(49_2)$. The term $1 - f(P, v)$ increases monotonically from 0 to 1 as $v$ ranges from $u$ to $\infty$, thus $U < k \int_u^\infty \frac{dv}{g(v)} \leq k \int_u^\infty \frac{dv}{(c^2+v)^{3/2}} = \frac{2k}{(c^2+u)^{1/2}}$. Hence $RU$ is bounded and of course $\lim\limits_{R\to\infty} U = 0$. Moreover from $(50_2)$

$$\left|\frac{\partial U}{\partial x}\right| \leq 2k|x| \int_u^\infty \frac{dv}{(a^2+v)(c^2+v)^{3/2}}$$

$$\leq 2k|x| \int_u^\infty \frac{dv}{(c^2+v)^{5/2}} = \frac{4}{3} \quad \frac{k\,|x|}{(c^2+u)^{3/2}}.$$

Since $(c^2+u)^{3/2}$ has the order $R^3$ and $\frac{|x|}{R} \leq 1$, it follows that

$R^2\left|\frac{\partial U}{\partial x}\right|$ is bounded. Likewise $R^2\left|\frac{\partial U}{\partial y}\right|$ and $R^2\left|\frac{\partial U}{\partial z}\right|$.

4. Method of electric images. Given a grounded plane conductor $S$ and a point charge $e$ at a point $Q$ outside the conductor. Charge is induced on the infinite plane. What is the potential $U$?

Take the plane $S$ as $(x, y)$-plane and $Q$ on the positive $z$-axis with coordinates 0, 0, $a$. From the theory of electricity it is known that $U = 0$ on the conductor, i.e. for $z = 0$, and also for $z < 0$. Let $Q'(0, 0, -a)$ be the image of $Q$ in $S$ and place the charge $-e$ at $Q'$. Then $(\rho^2 = x^2 + y^2)$

$$(54)\quad U = \frac{e}{QP} - \frac{e}{Q'P} = \frac{e}{\sqrt{\rho^2+(z-a)^2}} - \frac{e}{\sqrt{\rho^2+(z+a)^2}}$$

is the required potential at a point $P(x, y, z)$ of the half-space $T$ with $z \geq 0$: we have obviously $U=0$ on $S$, because $QP = Q'P$ if $P$ lies on $S$.

Find the density $\sigma$ of the induced charge at any point $P_0(x_0, y_0, 0)$ of $S$. Use equation (21) in Art. 6, considering that $\frac{\partial U}{\partial n}$ coincides with $\frac{\partial U}{\partial z}$. Obviously $\frac{\partial U_-}{\partial z} = 0$, since $U$ vanishes identically for $z < 0$. On the other hand, calculate $\frac{\partial U_+}{\partial z}$ from (54) and infer

$$\sigma = -\frac{ea}{2\pi\sqrt{x_0^2 + y_0^2 + a^2}^3}.$$

5. Given two grounded parallel plane conductors $S_1$ and $S_2$ and a point charge $e$ at $Q$ between the planes. What is the potential? Take $S_1$ and $S_2$ as the planes $z = 0$ and $z = \frac{c}{2}$ respectively and $Q$ on the positive $z$-axis with coordinates 0, 0, $a$ $\left(0 < a < \frac{c}{2}\right)$. If we place the charge $-e$ at $Q'(0, 0, -a)$, the image of $Q$ in $S_1$, the potential will vanish on $S_1$, but not on $S_2$. If we place, at $(0, 0, c - a)$ and $(0, 0, c + a)$, the images of $Q$ and $Q'$ in $S_2$, the charges $-e$ and $e$ respectively, the potential of the four charges will vanish on $S_2$, but not on $S_1$. If we continue in this way the $z$-coordinates of the next images are $-c + a$, $-c - a$, then $2c - a$, $2c + a$, etc., and we obtain an infinite series of images with ever-increasing distances from $S_1$ and $S_2$. These distances range through all numbers $nc + a$, $nc - a$ $[n = 0, \pm 1, \pm 2, \ldots]$ and the point charges are $e$ and $-e$ respectively. The effects of these charges are ever decreasing with increasing distances. Accordingly, the required potential is given by

$$U = \sum_{n=-\infty}^{+\infty} \left(\frac{e}{\sqrt{\rho^2 + (z - nc - a)^2}} - \frac{e}{\sqrt{\rho^2 + (z - nc + a)^2}}\right),$$

where $\rho^2 = x^2 + y^2$.

Prove that this series converges absolutely at any point dif-

ferent from $Q$ and its images, that it converges uniformly in any closed region not containing such points. Infer from the uniform convergence that $U$ is a regular potential at any point except $Q$ and its images by using the theorem in Art. 7, Chapter VIII, after equation (36). This theorem is, of course, valid for Newtonian potentials also. Show that $u \doteq 0$ on $S_1$ and $S_2$. Find the density of the induced charge on $S_1$ and $S_2$, applying again equation (21) in Art. 6.

# CHAPTER VI

## RELATION OF POTENTIAL TO THEORY OF FUNCTIONS

### Art. 1. The Conjugate Potential

In this chapter we shall deal entirely with logarithmic potential, not with Newtonian potential. We will discuss the relations which exist between logarithmic potential and the theory of functions of a complex variable, relations for which there is no parallel in the theory of Newtonian potential. Our starting point is the equation ((29*) in Chapter III)

$$\oint_S \frac{\partial u}{\partial n}\, ds = 0,$$

where the curve $S$ is the entire boundary of the region $T$ of regularity of the potential $u$, which region we now assume to be simply-connected. If we let $C$ be any closed curve in $T$ without double points, composed of a finite number of pieces with continuously turning tangent, then we know that

$$\oint_C \frac{\partial u}{\partial n}\, ds = 0. \tag{1}$$

*For directed open curves, we will assume that the normal points to the right side of the curve, and consider ds to be always positive.* If $P_0$ and $P$ are any two points in $T$ connected by two curves $C_1$ and $C_2$, with continuous tangents, which together form a closed curve $C$ in $T$, then

$$\oint_C \frac{\partial u}{\partial n}\, ds = \int_{C_1} \frac{\partial u}{\partial n}\, ds + \int_{C_2} \frac{\partial u}{\partial n}\, ds = 0,$$

where the integration is from $P_0$ to $P$ on $C_1$ and from $P$ to $P_0$

on $C_2$. Reversing the direction so that we integrate from $P_0$ to $P$ on $C_2$, this becomes

$$\int_{C_1} \frac{\partial u}{\partial n}\, ds = \int_{C_2} \frac{\partial u}{\partial n}\, ds. \tag{2}$$

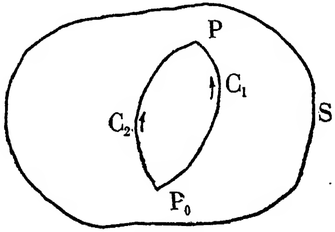


FIG. 8

The integral

$$\int_{P_0}^{P} \frac{\partial u}{\partial n}\, ds$$

then has the same value for all paths of the above type from $P_0$ to $P$. Hence the value of the integral depends merely on the position of the points $P_0$ and $P$, and not on the path used. If we hold the point $P_0$ fixed and consider $P:(x, y)$ as variable, the function $v(x, y)$ defined by

$$v = \int_{P_0}^{P} \frac{\partial u}{\partial n}\, ds \tag{3}$$

is a single-valued function of $(x, y)$ defined throughout $T$. Now

$$\frac{\partial u}{\partial n} = \boldsymbol{n} \cdot \nabla u = n_1 \frac{\partial u}{\partial x} + n_2 \frac{\partial u}{\partial y} = \cos \alpha_1 \frac{\partial u}{\partial x} + \cos \alpha_2 \frac{\partial u}{\partial y}, \tag{4}$$

where $\alpha_1, \alpha_2$ are the angles which $\boldsymbol{n}$ makes with the positive $x$ and $y$ axes. Also

$$\cos \alpha_1 ds = n_1 ds = dy, \quad n_2 ds = -dx, \tag{5}$$

so that

$$v = \int_{P_0}^{P} \left( \frac{\partial u}{\partial x} dy - \frac{\partial u}{\partial y} dx \right). \tag{6}$$

From this form of $v$, it is seen that

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{7}$$

To prove this, we form

$$v(x + h, y) = \int_{P_0}^{P'} \left( \frac{\partial u}{\partial x} dy - \frac{\partial u}{\partial y} dx \right),$$

where $P'$ is the point $(x + h, y)$ which is in $T$ if $h$ is sufficiently small. Now since our integral is independent of the path,

$$v(x+h, y) - v(x, y) = \left( \int_{P_0}^{P} + \int_{P}^{P'} - \int_{P_0}^{P} \right) \left( \frac{\partial u}{\partial x} dy - \frac{\partial u}{\partial y} dx \right)$$

$$= \int_{P}^{P'} \left( -\frac{\partial u}{\partial y} \right) dx = -h \cdot \frac{\partial u(x + \theta h, y)}{\partial y} \qquad [0 < \theta < 1]$$

by the mean value theorem for integrals, so that

$$\frac{\partial v}{\partial x} = \lim_{h \to 0} \frac{v(x + h, y) - v(x, y)}{h} = -\frac{\partial u}{\partial y}.$$

The other equation (7) is obtained in a similar manner. From (7) it follows that

$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y}, \qquad \frac{\partial^2 v}{\partial y^2} = +\frac{\partial^2 u}{\partial x \partial y};$$

and hence

$$\nabla^2 v = 0. \tag{8}$$

Moreover, from (7), $v$ has continuous first derivatives, and in fact we find that $v$ has continuous derivatives in $T$ of all orders, since $u$ does. Hence $v$ is a regular harmonic function in $T$; we call it the "*conjugate potential*" to $u$. It is defined by (7)

except for an additive constant, and completely defined by (3) or (6), with the condition $v(x_0, y_0) = 0$.

The potential conjugate to $v$ is $-u$, since

$$\frac{\partial(-u)}{\partial y} = \frac{\partial v}{\partial x}, \qquad \frac{\partial(-u)}{\partial x} = -\frac{\partial v}{\partial y}.$$

From (7), we have $\dfrac{\partial u}{\partial x} : \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial y} : \dfrac{\partial v}{\partial x}$. This is the condition that the curves $u$ = const. and $v$ = const. be perpendicular (the reader should prove this). Hence we can regard the lines $u$ = const. as the lines of force for the equipotential lines $v$ = const., or conversely we can regard the lines $v$ = const. as the lines of force for the potential $u$. We have previously defined the lines of force as the solutions of the differential equation $dy : dx = \dfrac{\partial u}{\partial y} : \dfrac{\partial u}{\partial x}$, leaving undetermined how this equation was to be solved. Now we are able to find the lines of force, and hence the solution of this differential equation, by integration of (6). In fact, the equation $v(x, y)$ = const., with $y$ regarded as an implicit function of $x$ containing an arbitrary constant, is the general solution of the differential equation. For the equation, on account of (7), becomes

$$dy : dx = -\frac{\partial v}{\partial x} : \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x}\,dx + \frac{\partial v}{\partial y}\,dy = 0, \; dv = 0, \; v = \text{const.}$$

The equations (7) are the equations which are well known in theory of functions as the *Cauchy-Riemann equations.* In the theory of functions it is proved that: *the real and imaginary parts $u$ and $v$ of an analytic function of a complex variable $z = x + iy$,*

(9) $$f(z) = u(x, y) + iv(x, y),$$

*satisfy* (7) *and therefore are conjugate functions; and conversely, the combination $u + iv$ of a harmonic function and its conjugate is an analytic function of $z$.* The region of regularity of $f(z)$ is the

region $T$ of regularity of $u$ and $v$. It is this theorem on which the relation between potential theory and theory of functions is based. We shall sketch the proof of the theorem. A function $f(z) = u + iv$ is called an analytic function, regular in the region $T$, if it has a continuous derivative

$$\lim_{\Delta z \to 0} \frac{\Delta f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = f'(z)$$

at every point of $T$. The limit of the difference quotient is assumed to exist regardless of the way in which $\Delta z = \Delta x + i\Delta y$ approaches 0.

Now let $f(z)$ be an analytic function. First put $\Delta y = 0$, so that $\Delta z = \Delta x$, and let $\Delta x \to 0$. Then

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) + iv(x + \Delta x, y) - u(x, y) - iv(x, y)}{\Delta x}$$

$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \cdot$$

Secondly put $\Delta x = 0$, $\Delta z = i\Delta y$, and let $\Delta y \to 0$. Then analogously

$$f'(z) = \frac{1}{i}\left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y}\right) \cdot$$

Therefore
$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y},$$

from which the Cauchy-Riemann equations follow.

Conversely, assume these equations to be valid. We easily get, according to the rules of the calculus, $\dfrac{\Delta f(z)}{\Delta z}$

$$= \frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - u(x, y) - iv(x, y)}{\Delta x + i\Delta y}$$

$$= \frac{\dfrac{\partial u}{\partial x}\Delta x + \dfrac{\partial u}{\partial y}\Delta y + i\left(\dfrac{\partial v}{\partial x}\Delta x + \dfrac{\partial v}{\partial y}\Delta y\right)}{\Delta x + i\Delta y} + \sigma$$

where $\lim \sigma = 0$, as $\Delta x$ and $\Delta y$ approach 0 or as $\Delta z = \Delta x + i\Delta y$ approaches 0 in any way. (The reader should prove this formula!) It follows on account of the Cauchy-Riemann equations that

$$\frac{\Delta f(z)}{\Delta z} = \frac{\frac{\partial u}{\partial x}(\Delta x + i\Delta y) + i\frac{\partial v}{\partial x}(\Delta x + i\Delta y)}{\Delta x + i\Delta y} + \sigma$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + \sigma.$$

Also $$\frac{\Delta f(z)}{\Delta z} = \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) + \sigma.$$

Consequently $\dfrac{\Delta f(z)}{\Delta z}$ has a limit as $\Delta z \to 0$ in any way, and $f(z)$ is therefore an analytic function. This limit is, of course,

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

All logarithmic potentials are obtained (in conjugate pairs) by taking the real and imaginary parts of all analytic functions of $z$. This is a very simple method of forming logarithmic potentials. For example

$$z = x + iy,\ z^2 = (x^2 - y^2) + i(2xy), \frac{1}{z} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2},$$

$$e^z = e^x \cos y + ie^x \sin y.$$

Also, after introducing polar coordinates

$$x = r\cos\theta,\ y = r\sin\theta,\ r = \sqrt{x^2+y^2},\ \theta = \arctan\frac{y}{x}$$

an important example is

$$\log z = \log r + i\theta$$

so that arc tan $(y/x)$ is the function conjugate to $\log\sqrt{x^2+y^2}$.

## Art. 2. Expansion in Cartesian and Polar Coordinates

Since every potential function can be regarded as the real part of an analytic function, it is not difficult to prove in a new manner the power series expansion previously derived. The new proof, however, assumes some knowledge of the theory of analytic functions and is only applicable to logarithmic potential. The function $f(z)$ whose real part is the given potential $u$ can be expanded in the neighbourhood of any regular point $z_0$ in a power series

$$f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n,$$

with constant coefficients $c_n$, which is known to converge in the largest circle $|z - z_0| < k$ which can be drawn in $T$ about $z_0$. To separate this into real and imaginary parts, we set

$$z = x + iy,\ z_0 = x_0 + iy_0,\ c_n = a_n + ib_n,$$

and hence

$$\begin{aligned} c_n(z - z_0)^n &= (a_n + ib_n)(x - x_0 + i(y - y_0))^n \\ &= (a_n + ib_n)\Big\{(x - x_0)^n - \binom{n}{2}(x - x_0)^{n-2}(y - y_0)^2 + \dots \\ &\quad + i\Big[\binom{n}{1}(x - x_0)^{n-1}(y - y_0) - \binom{n}{3}(x - x_0)^{n-3}(y - y_0)^3 + \dots\Big]\Big\}; \end{aligned}$$

therefore, if we denote the real part of a complex number $g$ by $\mathfrak{R}\,[g]$,

$$\begin{aligned} u = \mathfrak{R}\,[f(z)] = \sum_{n=0}^{\infty}\Big[ a_n\Big\{(x - x_0)^n - \binom{n}{2}(x - x_0)^{n-2}(y - y_0)^2 + \dots\Big\} \\ - b_n\Big\{\binom{n}{1}(x - x_0)^{n-1}(y - y_0) - \binom{n}{3}(x - x_0)^{n-3}(y - y_0)^3 + \dots\Big\}\Big]. \end{aligned}$$

It requires proof that we may remove the brackets in this expansion and rearrange the terms. It is sufficient to show that the double series in $x - x_0$, $y - y_0$ arrived at in this manner is absolutely convergent. Now

$$|a_n|\left\{|x-x_0|^n+\binom{n}{2}|x-x_0|^{n-2}|y-y_0|^2+\ldots\ldots\right\}$$
$$+|b_n|\left\{\binom{n}{1}|x-x_0|^{n-1}|y-y_0|+\binom{n}{3}|x-x_0|^{n-3}|y-y_0|^3+\ldots\right\}$$
$$<|c_n|\left\{|x-x_0|^n+\binom{n}{1}|x-x_0|^{n-1}|y-y_0|+\ldots\ldots\right\}$$
$$=|c_n|(|x-x_0|+|y-y_0|)^n.$$

The series $\sum_{n=0}^{\infty}|c_n|(|x-x_0|+|y-y_0|)^n$ converges for $|x-x_0|+|y-y_0|<k$ since the series $\sum_{n=0}^{\infty}|c_n||z-z_0|^n$ converges for $|z-z_0|<k$. Therefore the double series converges absolutely, also, for $|x-x_0|+|y-y_0|<k$, and hence certainly when $|x-x_0|<\frac{k}{2}$ and $|y-y_0|<\frac{k}{2}\cdot$ Thus the convergence of the resulting Taylor's series

$$u(x, y)=\sum_{i,j=0}^{\infty}c_{ij}(x-x_0)^i(y-y_0)^j$$

is established for a sufficiently small neighbourhood about $(x_0, y_0)$.

We now assume that $(x_0, y_0)=(0, 0)$ for simplicity, and introduce polar coordinates $x = R\cos\phi$, $y = R\sin\phi$, giving

$$f(z)=\sum_{n=0}^{\infty}c_nR^ne^{n\phi i}=\sum_{n=0}^{\infty}(a_n+ib_n)R^n(\cos n\phi+i\sin n\phi)$$
$$u=\sum_{n=0}^{\infty}R^n(a_n\cos n\phi-b_n\sin n\phi)$$
$$v=\sum_{n=0}^{\infty}R^n(b_n\cos n\phi+a_n\sin n\phi).$$

The expansions of $u$ and $v$, like that of $f(z)$, are valid in the largest circle about the origin which lies in the region of regularity $T$.

We may note that if $u$ and its derivatives with respect to $x$ and $y$ to the $m^{th}$ order vanish at the origin, the coefficients $c_{ij}$ of the Taylor series vanish for $i + j \leq m$, and accordingly the coefficients $a_n$ and $b_n$ vanish for $n \leq m$.

## Art. 3. Converse of the Ideas in Art. 1

In Art. 1 we obtained the function $v$ conjugate to a given harmonic function, and showed that these functions satisfy the Cauchy-Riemann equations. We will now reverse the procedure and prove that: *If two functions $u$ and $v$ are continuous, with continuous first partial derivatives, in a bounded simply-connected region $T$ and satisfy the Cauchy-Riemann equations there, then they are regular potential functions in $T$.* This theorem evidently permits the use of the concept of regular potential functions without making use of second derivatives (compare Chapter II, Art. 7).

For the proof we form the integral

$$\oint_{C'} (udx - vdy)$$

over any closed curve $C'$ free of double points, with continuously turning tangent and lying entirely within $T$. If we call $T'$ the region bounded by $C'$, then (see Chapter III, Art. 1)

$$\oint_{C'} (udx - vdy) = -\iint_{T'} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dS.$$

From one of the Cauchy-Riemann equations, the integrand of the area integral vanishes identically, so that

$$\oint_{C'} (udx - vdy) = 0. \tag{10}$$

Since the integral over every closed curve vanishes, it follows that

$$\int_{P_0}^{P} (udx - vdy) \tag{11}$$

is independent of the path of integration as long as this lies in $T$, and hence depends on the end-points only. With $P_0$ fixed and $P$ variable, this defines a single-valued function of $(x, y)$ throughout $T$. This function, which we designate by $F(x, y)$, has the properties that (see Art. 1)

$$\frac{\partial F}{\partial x} = u, \quad \frac{\partial F}{\partial y} = -v, \tag{12}$$

and hence

$$\frac{\partial^2 F}{\partial x^2} = \frac{\partial u}{\partial x}, \quad \frac{\partial^2 F}{\partial y^2} = -\frac{\partial v}{\partial y}$$

so that on account of the other Cauchy-Riemann equation,

$$\Delta^2 F = 0. \tag{13}$$

Therefore the function $F(x, y)$ possesses continuous second derivatives in $T$ and satisfies Laplace's equation there. Thus $F(x, y)$ is a regular potential function in $T$. Accordingly its derivatives, and hence the functions $u$ and $v$, are likewise regular harmonic functions in $T$.

*Note.* The condition of continuous first derivatives can be replaced by a milder one: If $u$ and $v$ are continuous in $T$ and possess there first partial derivatives satisfying the Cauchy-Riemann equations, and if they also have the property of "complete differentiability," then $u$ and $v$ are regular potential functions in $T$. The hypothesis of complete differentiability for the function $w$ is that we can write

$$\Delta w = w(x + \Delta x, y + \Delta y) - w(x, y) = \frac{\partial w}{\partial x}\Delta x + \frac{\partial w}{\partial y}\Delta y + \epsilon,$$

where $\epsilon$ is an infinitesimal of higher order than $\Delta x$ and $\Delta y$ in the sense that

$$\frac{\epsilon}{|\Delta x| + |\Delta y|} \to 0 \text{ as } \Delta x \to 0, \Delta y \to 0.$$

This condition is certainly fulfilled if the partial derivatives are continuous, but it may be fulfilled when the derivatives are not continuous.

With respect to the proof, we merely note briefly that the hypotheses on $u$ and $v$ assure the existence of a finite derivative $f'(z)$ of the function $f(z) = u + iv$.[1] Then, by application of a famous theorem of Goursat[2], it follows that $f$ is an analytic function. From this our theorem follows immediately.

## Art. 4. Invariance of Potential under Conformal Mapping

If $X + iY = f(z) = f(x + iy)$ is an analytic function of $z$, which is regular in the neighbourhood of a point $z_0$ and has a non-vanishing derivative there, i.e. $f'(z_0) \neq 0$, then it is known (Chapter VIII, Art. 12) from the theory of functions that the neighbourhood of $z_0$ is mapped by this function conformally (i.e., with preservation of angles) in a one-to-one manner on the neighbourhood of $Z_0 = f(z_0)$. The inverse function $z = F(Z)$ is analytic, regular in the neighbourhood of $Z_0$. If we wish to remain in the field of real variables, we can state these results as follows: If the functions $X(x, y)$ and $Y(x, y)$ have continuous first partial derivatives in the neighbourhood of a point $(x_0, y_0)$ which satisfy the Cauchy-Riemann equations

$$\frac{\partial X}{\partial x} = \frac{\partial Y}{\partial y}, \quad \frac{\partial X}{\partial y} = -\frac{\partial Y}{\partial x}, \tag{14}$$

and if their Jacobian

$$D = \begin{vmatrix} \dfrac{\partial X}{\partial x} & \dfrac{\partial X}{\partial y} \\[2ex] \dfrac{\partial Y}{\partial x} & \dfrac{\partial Y}{\partial y} \end{vmatrix} \neq 0 \tag{15}$$

in this region, then the equations

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[1] See, for example, Osgood, Funktionentheorie, 3.Aufl., 6.Kap. §6.

[2] See, for example, B. K. Knopp, Functionentheorie I, §19, theorem 3.

$$(16) \qquad X = X(x, y), \quad Y = Y(x, y)$$

map the region about $(x_0, y_0)$ in a one-to-one, reversible, and conformal manner on the region about the point $X_0 = X(x_0, y_0)$, $Y_0 = Y(x_0, y_0)$ of the $(X, Y)$-plane. The functions (16) and the inverse functions possess continuous derivatives of all orders in the neighbourhood of $(x_0, y_0)$ and $(X_0, Y_0)$ respectively. By combining (14) and (15), we find besides that

$$D = \left(\frac{\partial X}{\partial x}\right)^2 + \left(\frac{\partial Y}{\partial x}\right)^2.$$

Now let $u$ be a regular potential in the neighbourhood of $(x_0, y_0)$. By the equations inverse to (16), $u$ is defined as a function of $X$, $Y$ which has continuous derivatives of all orders in the neighbourhood of $(X_0, Y_0)$. We will show that $u$, regarded as a function of $X$, $Y$, is also a potential, since it satisfies Laplace's equation. Now

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y}\frac{\partial Y}{\partial x}, \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y}\frac{\partial Y}{\partial y},$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial X^2}\left(\frac{\partial X}{\partial x}\right)^2 + 2\frac{\partial^2 u}{\partial X \partial Y}\frac{\partial X}{\partial x}\frac{\partial Y}{\partial x} + \frac{\partial^2 u}{\partial Y^2}\left(\frac{\partial Y}{\partial x}\right)^2$$
$$+ \frac{\partial u}{\partial X}\frac{\partial^2 X}{\partial x^2} + \frac{\partial u}{\partial Y}\frac{\partial^2 Y}{\partial x^2}, \quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial X^2}\left(\frac{\partial X}{\partial y}\right)^2 + \cdots\cdots\cdots$$

From this it follows, after making use of (14), that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \left\{\left(\frac{\partial X}{\partial x}\right)^2 + \left(\frac{\partial Y}{\partial x}\right)^2\right\}\left(\frac{\partial^2 u}{\partial X^2} + \frac{\partial^2 u}{\partial Y^2}\right) = D\left(\frac{\partial^2 u}{\partial X^2} + \frac{\partial^2 u}{\partial Y^2}\right).$$

Since $D \neq 0$, it follows from $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ that $\dfrac{\partial^2 u}{\partial X^2} + \dfrac{\partial^2 u}{\partial Y^2} = 0$. The theorem which we have just proved can be stated: A

regular potential function is invariant under a conformal transformation.

The theorem of this Art. is very simple, when expressed in terms of analytic functions of complex variables. Let $u + iv = g(z) = g(x + iy)$ be analytic regular at $z_0 = x_0 + iy_0$, so that $u(x, y)$ is harmonic regular at $(x_0, y_0)$; and suppose $Z = X + iY = f(z)$ and its inverse $z = F(Z)$ are analytic regular at $z_0$ and $Z_0 = f(z_0)$ respectively. (This implies that $f'(z_0) = 1/F'(Z_0) \neq 0$.) Hence $g(z) = g(F(Z)) = U + iV$ is analytic regular at $Z_0$ and therefore $U(X,Y)$ harmonic regular at $(X_0, Y_0)$; moreover, obviously, $u(x, y) = U(X, Y)$.

# CHAPTER VII

## THE BOUNDARY VALUE PROBLEMS OF POTENTIAL THEORY

### Art. 1. Statement of the Problems

We have already stated that it is not possible to prescribe arbitrarily both the value of the potential and of its normal derivative on the entire boundary of the region of regularity. We will soon see that a potential function is uniquely determined if *either* the value of the function *or* the value of its normal derivative is given on the complete boundary of a region. We must first formulate more exactly the problems which arise in this manner.

Let $R$ be a finite region in space, bounded by $S$. We will assume that $S$ is a closed surface with a continuously turning normal, in general; however, it may have a finite number of corners and conical points. Since multiply-connected regions are to be allowed, $S$ may consist of a finite number of separate non-intersecting surfaces. Let $f$ be a function defined and continuous at all points of $S$. Then the *first boundary value problem* may be stated as follows: *to determine a solution $u$ of Laplace's equation $\nabla^2 u = 0$ which is regular in $R$ and continuous in $R + S$, and takes on the prescribed values $f$ on $S$.* The last condition means that on approaching $S$ from the interior, $u$ converges to $f$. This first boundary value problem is also called the "*Dirichlet problem.*"

The *second boundary value problem* of potential theory, or "*Neumann problem*", is: *to find a solution $u$ of Laplace's equation which is regular in $R$ and which, along with its normal derivative, is continuous in $R + S$, and whose normal derivative approaches the given function $f$ on $S$.* Since we know that

$\iint \frac{\partial u}{\partial n} dS = 0$ for any closed surface inside which $u$ is harmonic, it follows that the function $f$ here is not completely arbitrary, but must satisfy the condition

$$\iint_S f dS = 0.$$

In the theory of logarithmic potential, the first and second boundary value problems are stated in exactly similar form. The boundary $C$ is a closed curve consisting of a finite number of pieces with continuously turning tangent; or $C$ may consist of a finite number of separate non-intersecting curves of the above kind.

Boundary value problems arise in theoretical physics; in general, the first boundary value problem arises in electrostatics and in heat conduction, while the second boundary value problem is met in hydrodynamics. For example, in the theory of heat conduction, it is proved that the temperature $u$ of a body, if it is independent of the time (i.e., for the so-called stationary state), satisfies Laplace's equation and hence is a potential in our general sense. Here the temperature of the surface of the body may be arbitrarily given as a function of the position on the surface.

Also, in the stationary flow of an incompressible fluid, a "velocity potential" may exist (see end of Art. 5 in Chapter III). This is a function whose vector gradient is the fluid velocity. This function also satisfies Laplace's equation. The rate of flow outward through an element of surface is proportional to $\frac{\partial u}{\partial n}$ and may be arbitrarily prescribed. But it is necessary that the function $f = \frac{\partial u}{\partial n}$ satisfies the condition $\iint_S \frac{\partial u}{\partial n} dS = 0$. Physically, this is the condition for incompressibility (Chapter III, equation (12)).

When the stationary flow of heat in a body is investigated, assuming that the flow through the surface is governed by Newton's law of cooling, the value of the quantity

$$k\frac{\partial u}{\partial n} + hu$$

is given on the surface. Accordingly, we formulate the third boundary value problem.

We seek a potential which satisfies the surface condition $k\frac{\partial u}{\partial n} + hu = f$ where $k$, $h$ and $f$ are arbitrary prescribed functions on $S$ (continuous functions). In the above case $k$ and $h$, the conductivity in the interior and that through the surface, are positive. The case where $k$ and $h$ are of opposite signs is of interest in hydrodynamics. The first and second boundary value problems are special cases of the third for $k = 0$ or $h = 0$. We will not be as much interested in the third boundary value problem as in the first and second.

We will consider uniqueness theorems in this chapter. In Chapter VIII we will prove the existence of the solution of the first boundary value problem for the circle by means of the Poisson integral, and in Chapter IX we will consider the corresponding problems in space. In the last two chapters we will develop the Fredholm theory of integral equations and apply them to the boundary value problems of potential theory.

Dirichlet was the first to attempt a general existence proof. His method, called the "Dirichlet principle" by Riemann, was later proved by Weierstrass to be inexact. Later, C. Neumann, H. Poincaré, H. A. Schwarz, E. R. Neumann, Fubini, Lebesgue, Zaremba have given strict proofs. Further, the boundary value problems have been solved in a very elegant manner by J. Fredholm by means of integral equations. Hadamard's determinant theorem plays an important role in Fredholm's

theory. Moreover D. Hilbert and R. Courant have reformulated the Dirichlet principle in a form free from objection, and O. Perron has proved the existence theorem in an entirely new manner.

## Art. 2. Uniqueness Theorems

*The first boundary value problem has at most one solution.*

For let $u_1$ and $u_2$ be two solutions; then it is to be proved that $v = u_1 - u_2$ vanishes identically in $R + S$. Evidently $v$ is a potential function regular in $R$ and continuous in $R + S$, which vanishes at all points of $S$. Assume that $v$ is different from zero at some point in $R$, hence either positive or negative there. Then $v$ must have a maximum or minimum in $R$, which is impossible from Chapter III, Art. 8. Hence there can be no point in $R$ where $v$ is different from zero, so that we conclude that $u_1 \equiv u_2$. This proof is valid for Newtonian potential as well as for logarithmic.

*The second boundary value problem has at most one solution, except for an additive constant.*

Let $u_1$ and $u_2$ be two solutions; then we have to prove that $v = u_1 - u_2$ remains constant in $R$. It is evident that $v$ is also a regular potential in $R$, and that $v$ and $\dfrac{\partial v}{\partial n}$ remain continuous on approaching the boundary $S$, and that $\dfrac{\partial v}{\partial n} = 0$ on $S$. We can therefore say that $v$ is a solution of the second boundary value problem for the special case $f = 0$. But from the first form of Green's formula, now assuming Newtonian potential (the case of logarithmic potential is handled similarly), we have

$$\iiint (\text{grad } v)^2 \, dV = \iint v \frac{\partial v}{\partial n} \, dS;$$

and hence, on account of the boundary condition $\frac{\partial v}{\partial n} = 0$,

$$\iiint_R \left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right] dV = 0.$$

Noting that the integrand here is necessarily non-negative and is continuous, it follows that it must be zero everywhere in $R$, so that we must have $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial v}{\partial z} = 0$, from which it follows that $v$ is a constant.

*The third boundary value problem has at most one solution, if* $\frac{h}{k} > 0$.

If we let $v = u_1 - u_2$ again, then $\frac{\partial v}{\partial n} + \frac{h}{k} v = 0$ on $S$. Then from Green's formula again,

$$\iiint_R (\text{grad } v)^2 dV = \iint_S v \frac{\partial v}{\partial n} dS = -\iint_S \frac{h}{k} v^2 dS.$$

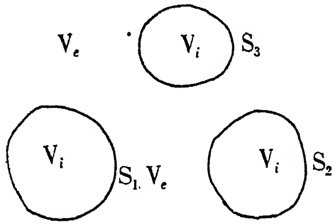
Here we note that $\frac{h}{k} > 0$ everywhere. Since a positive quantity cannot equal a negative one, it follows that both the surface and the space integrands must vanish identically in their integration regions. Hence grad $v = 0$, so that $v = \text{const.}$ in $R$, and $v = 0$ on $S$; hence by continuity $v = 0$ in $R + S$, so that $u_1 = u_2$ in $R + S$.

It should be noted that Green's formula, on which the proof for uniqueness of the second and third boundary value problems depends, cannot be used on the first boundary value problem. For the proof of Green's formula requires that $\frac{\partial v}{\partial n}$ as well as $v$ approach limits when the boundary is approached;

but in the first boundary value problem only the continuity of $v$ and not that of $\frac{\partial v}{\partial n}$ was assumed.

**Art. 3. The Exterior Problems**

The above boundary value problems are known as interior problems, in contrast with the "exterior" problems to which we now turn. In the interior problems, a function was sought which was harmonic in the interior of $S$, which we will now designate by $V_i$, and which satisfied certain linear boundary conditions when the field-point approached $S$ from the interior.




3 closed surfaces ($m = 3$)

FIG. 9

In the exterior boundary value problems, the corresponding problems are treated for the region $V_e$, exterior to $S$, which contains the infinitely distant points of the plane or space. We shall often follow the usage common in theory of functions, and speak of the region outside a very large circle as the neighbourhood of "the point at infinity" or the "infinite point," and say that a point which is becoming infinitely distant from the origin is approaching the point at infinity. This is natural because of the frequent use of inversion, in which the regions around the origin and around the "point at infinity" are interchanged.

The desired harmonic function is required to be regular in the exterior region including the point at infinity. We assume that the boundary may consist of several closed surfaces instead of only one, designated by $S_1, S_2, \ldots\ldots, S_m$. We designate the entire set of boundary surfaces by $S$. And the region $V_e$ designates the region which is composed of all points lying outside all the $S_i$, or in other words, outside of $S$.

We will formulate the exterior Dirichlet problem for the region $V_e$ outside $S$ as follows: *to find a harmonic function of the form $u = w + c$, where $w$ is regular in $V_e$ and continuous in $V_e + S$ and $c$ is a constant, such that we have $u = f$ on $S$ and $w$ has the mass $M$, where $f$ and $M$ are prescribed.* The uniqueness of the solution is proved as follows: let $u_1 = w_1 + c_1$ and $u_2 = w_2 + c_2$ be two solutions, and let $v = u_1 - u_2 = w_1 - w_2 + C$ $(C = c_1 - c_2)$. Then $v$ takes the value 0 on $S$. The harmonic function $w_1 - w_2$ is regular in $V_e$ and continuous in $V_e + S$, takes the constant value $-C$ on $S$ and has the mass 0 (because $w_1$ and $w_2$ have the same mass). Now, since a regular harmonic function with mass 0 can have no maximum or minimum at infinity and no maximum or minimum anywhere in $V_e$ (Chapter IV, Art. 8), it follows that the identity $w_1 - w_2 = -C$ must hold in $V_e + S$. Since also $w_1 = 0$, $w_2 = 0$ at infinity, it follows that $C = 0$ or $c_1 = c_2$ and $w_1 = w_2$ everywhere in $V_e + S$.

The problem can also be stated in the following form: *to find a harmonic function $u$ which is regular in $V_e$ and continuous in $V_e + S$ and takes on the value $f$ on $S$.* The uniqueness of the solution of the problem thus formulated, and its existence, will be proved in Chapter XI, Art. 3.

In the exterior Neumann (or second boundary value) problem, a harmonic function $u$ is sought which is regular in $V_e$ and continuous with continuous normal derivative in $V_e + S$, and satisfies the condition $\dfrac{\partial u}{\partial n} = f$ on $S$. In the third boundary value problem, the corresponding condition is

$$\frac{\partial u}{\partial n} + \frac{h}{k} u = f.$$

In both these problems, the uniqueness is proved just as in the corresponding interior problems, since the Green's formula used there is valid for $V_e$.

We emphasize the following remark: in the interior problem, the use of several separate surfaces as the boundary is no generalization, as the interiors are several unconnected regions; hence here it is sufficient to consider a single boundary surface $S$.

The three corresponding problems for logarithmic potential may be treated in a similar manner.

## Art. 4. The Dirichlet Principle. Direct Methods of Calculus of Variations

We will now investigate further the Dirichlet principle mentioned in Art. 1. The first boundary value problem is very closely connected with the Dirichlet variational problem: to make the "Dirichlet integral"

$$(1) \quad D = D(w) = \iiint_V \left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] dV = \iiint_V (\nabla w)^2 dV$$

a minimum. More precisely, what function, among all the functions $w$ which have continuous second derivatives in $V+S$ and take on the prescribed continuous values $f$ on $S$, gives the smallest value for the integral $D$? The same hypotheses as before are assumed on the nature of $V$ and $S$.

Assume that the problem has a solution $u$. Then for all "admissible" functions $w$, i.e. for all functions $w$ (different from $u$) which satisfy the above continuity and boundary conditions, we must have $D(w) > D(u)$. If $v$ also satisfies the continuity conditions and vanishes on $S$, then $w = u + \lambda v$, where $\lambda$ is an

arbitrary constant parameter, is an admissible function. We therefore form $D(u + \lambda v)$, and, using the abbreviation

$$(2)\quad D(u, v) = \iiint\limits_V \left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\right] dV = \iiint\limits_V \nabla u \cdot \nabla v \, dV$$

for convenience, find that

$$(3)\qquad D(u + \lambda v) = D(u) +$$

Evidently $D(u, u) = D(u)$. The expression $2\lambda D(u, v)$ is known in the calculus of variations as the "first variation" of $D(u)$. Now it may be seen to be necessary that for every function $v$ of the form described,

$$(4)\qquad D(u, v) = 0.$$

Since if $D(u, v) \neq 0$, we could choose the absolute value of $\lambda$ so small that the value of $2\lambda D(u, v) + \lambda^2 D(v)$ would have the same sign as its first term, and then choose the sign of $\lambda$ so that $\lambda D(u, v) < 0$. Then we would have $D(u + \lambda v) < D(u)$, so that $u$ is not a solution of the Dirichlet problem, which is a contradiction.

From (4) by the use of the first Green's formula (Chapter III, equation (25)), we get

$$D(u, v) = -\iiint\limits_V v\nabla^2 u\, dV + \iint\limits_S v\frac{\partial u}{\partial n}\, dS,$$

and because $v = 0$ on $S$,

$$(5)\qquad \iiint\limits_V v\nabla^2 u\, dV = 0.$$

From this, we conclude that

$$(6)\qquad \nabla^2 u = 0$$

throughout $V$. For if $\nabla^2 u$ were not zero, but say positive at some point in $V$, then on account of its continuity it would be positive in some neighbourhood, for example in a small sphere,

about this point. Then it would be possible so to define a function $v$ that it would be positive inside this sphere and zero outside it while retaining the required continuity properties even at the surface of this small sphere; for this function $v$, we would have

$$\iiint v\nabla^2 u\, dV > 0,$$

which is a contradiction to (5). We see therefore that any solution of the Dirichlet variational problem is also a solution of the first boundary value problem.

Moreover, it is sufficient to assume that $u$, $v$, and $w$ merely have continuous derivatives to the second order in $V$ and are continuous in $V + S$, while nothing is assumed about the derivatives on approaching $S$ except that they are uniformly bounded. For if $S'$ is a neighbouring surface lying just inside $S$, we may apply the Green's theorem to the region $V'$ inside $S'$, giving

$$\iiint_{V'} (\nabla u \cdot \nabla v)\, dV = -\iiint_{V'} v\nabla^2 u\, dV + \iint_{S'} v\frac{\partial u}{\partial n}\, dS.$$

Here the surface integral may be made arbitrarily small by letting $S'$ approach $S$, since $v$ has a value uniformly close to 0 on $S'$ and $\frac{\partial u}{\partial n}$ is uniformly bounded there. Moreover, the integral $\iiint_{V} \nabla u \cdot \nabla v\, dV$ differs arbitrarily little from $\iiint_{V} \nabla u \cdot \nabla v\, dV$ and also on account of (4) is arbitrarily near 0. Hence $\iiint_{V'} v\nabla^2 u\, dV$ is arbitrarily small and also arbitrarily near in value to $\iiint_{V} v\nabla^2 u\, dV$. But since this last integral has a value independent of the passage to the limit, we conclude again that (5), and therefore (6), must hold.

Now assume, conversely, that $u$ is a solution of the first boundary value problem. Then for every function $v$ having the required continuity properties and vanishing on $S$, we have by the Green's formula that $D(u, v) = 0$. Hence $D(u+v) = D(u)+D(v)$. But it is evident that $D(v) \geq 0$, and is only $= 0$ when $v$ is a constant and therefore zero throughout $V + S$ since $v$ is continuous and vanishes on $S$. Hence $D(u + v) > D(u)$ unless $v = 0$. We see therefore that a solution of the first boundary value problem is also a solution of the Dirichlet variational problem.

*The Dirichlet variational problem and the first boundary value problem are therefore equivalent problems.*

They have the same solution, in case such a solution exists. (It has been proved, Art. 2, that not more than one solution exists.)

The integrals $D(w)$ formed for admissible functions $w$ are a set of numbers which has a lower bound, since $D(w) > 0$. From this fact Riemann concluded that a function $u$ must exist, which makes the integral a minimum. This method of reasoning is called the "*Dirichlet principle.*" However, the conclusion is not valid. Its falsity consists in a failure to distinguish between "lower bound" and "minimum," an error which led to some false results in early mathematical history. An infinite set of numbers which has a lower bound certainly has a lower limit $m$, which has the properties that all numbers of the set (except a finite number of them) are $> m$, but that there are always numbers of the set $< m + \epsilon$ when $\epsilon$ is any positive constant, no matter how small. Thus the set of numbers $\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$ has the lower limit 0. On the other hand, such a number set may have no minimum; in this example the number 0 does not belong to the set. The numbers $D(w)$ likewise certainly have a lower limit; but from this

it does not follow that a function $u$ exists, for which $D(u)$ is equal to this lower limit, or which makes $D$ a minimum.

Weierstrass discovered the error in Riemann's method of reasoning, and gave an example also where no solution existed. However, Hilbert[1] later brought the Dirichlet principle back into good repute, by showing that under certain limiting conditions on the boundary and the prescribed boundary values the calculus of variations problem actually does have a solution. (This accordingly is also a solution of the first boundary value problem.)

If the boundary value problem is solved in any manner, as for example by the use of integral equations, then the variational problem is thereby solved also. Or, the latter problem can be solved directly, as Hilbert did, thereby solving the boundary value problem also. We will consider a few remarks about the direct methods of the calculus of variations. The variational problem may be reduced to differential equations (Euler's equations), and then an attempt may be made to solve or integrate these differential equations. This is the well known method used in the early calculus of variations. More recently, however, the "direct methods" have come to the fore, in which an attempt is made to solve the problem without reducing it to a problem in differential equations. Moreover, boundary value problems in partial differential equations may be transformed into variational problems; so that the corresponding Euler equations are the given differential equations, and then the newer direct methods are applied.

We will briefly sketch the direct method here in connection with the Dirichlet problem. Since the numbers $D(w)$ have a lower limit $m$, there certainly exist sequences of admissible functions $w_1, w_2, \ldots$ such that $D(w_n) \to m$ as $n \to \infty$. This is known from the elementary theory of real functions. Such a sequence of functions is known as a "minimal sequence."

[1]*Journal f. Mathematatik*, Bd. 129, 1905, and *Math. Annalen*, Bd. 59, 1905.

But it is not certain, even if a solution $u$ of the calculus of variations problem exists, that $w_n \to u$. Therefore it is necessary to select from the sequence $w_n$ a sub-sequence $g_n$, so that we actually have $g_n \to u$. To obtain such a sequence, we may proceed as follows. Let the region $V$ be covered, or filled out, by a denumerable infinity of spheres $C_i(i = 1, 2, 3, \ldots)$. The number of the spheres containing any given point inside $V$ is finite, but increases without bound as $S$ is approached. For the sphere $C_1$ the first boundary value problem is solved, using the values of $w_n$ on its boundary. (This may be done by means of the Poisson integral described in Chapter IX.) Now let $g_n$ be that function which is the solution of the above boundary value problem in $C_1$ and is identical with $w_n$ in the remainder of $V$. The function $g_n$ is therefore harmonic inside $C_1$. The functions $g_n$ then form again a minimal sequence. For $D(g_n) \leq D(w_n)$ when the Dirichlet integral is taken over $C_1$, and $D(g_n) = D(w_n)$ when this integral is taken over the remainder of $V$. Hence $D(g_n) \leq D(w_n)$ when the integral is taken over all of $V$. Now since $D(w_n) \to m$, it follows that $D(g_n) \to m$ (for certainly it cannot happen that $D(g_n) < m$). Moreover, it can be easily proved that the minimal sequence $g_n$ converges inside $C_1$ to a harmonic function. Outside $C_1$ the sequence $g_n$ may not converge to a limiting function. This same method may be applied to the other spheres $C_2, C_3, \ldots$, leading to limiting functions which are harmonic in them. Furthermore it may be easily proved that all these harmonic functions represent the same function, since they are analytic continuations of each other (see Chapter IX). We obtain in this manner a harmonic function defined throughout $V$. Finally it may be proved that this function, like the functions $w_n$ and $g_n$ from which it was derived, takes on the prescribed value $f$ on the boundary $S$ of the region $V$. This concludes the existence proof.

# CHAPTER VIII

## THE POISSON INTEGRAL IN THE PLANE

### Art. 1. Solution of the Dirichlet Problem for the Circle

We will now solve the first boundary value problem for the interior of a circle. The solution of this problem is given explicitly by an integral over the boundary of the circle, which was found by Poisson.

Let $C$ be a circle about the origin $O$ of radius $l$, and let a continuous function $f$ be given on its boundary $S$ ($C$ is the *interior* of the circle). *The solution of this interior problem is*

$$u(x, y) = -\frac{1}{\pi}\oint_S f\left\{\frac{\cos(r, n)}{r} + \frac{1}{2l}\right\} ds, \tag{1}$$

where $\boldsymbol{r}$ is the vector distance to the field-point $P:(x, y)$ from the integration point $Q:(\xi, \eta)$ on the circumference $S$ of the circle, and $\boldsymbol{n}$ is the outward normal, and hence in the same direction as $OQ$.

To prove this, write $u$ in the form

$$u(x, y) = -\frac{1}{\pi}\oint_S \frac{f\cos(r, n)}{r}\, ds - \frac{1}{\pi}\int_S \frac{f}{2l}\, ds. \tag{1*}$$

The integral

$$W = \oint_S \frac{f\cos(r, n)}{r}\, ds$$

is the potential of a double distribution on $S$, and hence a regular potential function in the interior $C$. The second integral on the right side of (1*) is a constant, and hence certainly a regular harmonic function. Accordingly the function $u$ is a regular harmonic function in the interior region $C$. Since

$f$ is the moment of the double distribution, the potential $W$ on approaching the boundary point $A$ satisfies the condition

$$W_- = W_A - \pi f_A$$

(see Equation 45 of Art. 8, Chapter 5), where

$$W_A = \oint_S \frac{f \cos(r, n)}{r} ds$$

is the value of $W$ when $P$ is at the boundary point $A$. But for $P$ at $A$ on the circle $S$,

$$\cos(r, n) = -\cos(OQP) = \frac{-r}{2l}, \quad \frac{\cos(r, n)}{r} = -\frac{1}{2l}$$

so that
$$W_- = -\oint_S \frac{f}{2l} ds - \pi f_A$$

and accordingly, the limiting value of $u$ when $P$ approaches the boundary circle at $A$ is

$$u_- = \frac{1}{\pi}\int_S \frac{f\,ds}{2l} + f_A - \frac{1}{\pi}\int_S \frac{f\,ds}{2l} = f_A,$$

which concludes the proof that (1) solves the boundary value problem.

This integral can be put in a form better suited for applications. For this purpose, introduce polar coordinates with the centre of the circle as pole. Let $R$ and $\phi$ be the polar coordinates of $P$, and $l$ and $\psi$ be those of $Q$. Then

$$R^2 = l^2 + r^2 + 2lr\cos(r, n),$$
$$r^2 = l^2 + R^2 - 2lR\cos(\phi - \psi),$$
$$ds = l\,d\psi,$$

so that the integral (1) is easily seen to become

$$(2) \quad u = \frac{1}{2\pi}\oint_S f\frac{l^2 - R^2}{lr^2} ds = \frac{1}{2\pi}\oint_S \frac{(l^2 - R^2) f\,d\psi}{l^2 + R^2 - 2lR\cos(\phi -

(The quantity $l^2 - R^2$ may, of course, be brought out in front of the integral sign.) *The integral in this form was obtained by Poisson, and it is known as the Poisson integral.* It is of fundamental importance in potential theory; and will be of great value in the further developments.

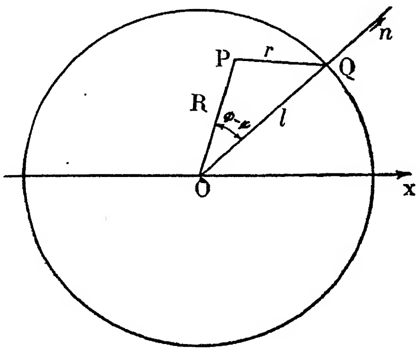



FIG. 10

The exterior boundary value problem for the circle is easily solved. The solution is

$$(3) \qquad u(x, y) = \frac{1}{\pi} \oint_S f \left\{ \frac{\cos(r, n)}{r} + \frac{1}{2l} \right\} dS$$

or

$$(4) \qquad u(x, y) = \frac{1}{2\pi} \oint_S f \frac{R^2 - l^2}{r^2} \, d\psi.$$

Formally, this differs only in sign from the solution for the interior of the circle. The proof is entirely similar to that for the interior. We emphasize that $u$ is regular also at the infinite point.

## Art. 2. Expansion in the Circle

We will now obtain the expansion of the solution of the problem for the interior of the circle, given by the Poisson integral, in the form of a series:

$$u(R, \phi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\phi + b_n \sin n\phi)R^n,$$

where the $a_n$ and $b_n$ are constants.

We will assume for simplicity that the circle is the unit circle. This can always be done through a transformation which is a mere change in scale: $x_1 = \frac{x}{l}$, $y_1 = \frac{y}{l}$. Since harmonic functions are evidently invariant under this transformation, this assumption is no restriction in generality. From equation (2) of the preceding paragraph, we have then

$$(5) \qquad u(R, \phi) = \frac{1}{2\pi}\int_0^{2\pi} f(\psi)\frac{1 - R^2}{1 + R^2 - 2R\cos(\phi - \psi)}\,d\psi.$$

Let

$$z = x + iy = Re^{\phi i} = R(\cos\phi + i\sin\phi);$$

then it is easily seen that $u$ is the real part of

$$(6) \qquad F(z) = \frac{1}{2\pi}\int_0^{2\pi} f(\psi)\frac{e^{\psi i} + z}{e^{\psi i} - z}\,d\psi.$$

For

$$\frac{e^{\psi i} + z}{e^{\psi i} - z} = \frac{\cos\psi + R\cos\phi + i(\sin\psi + R\sin\phi)}{\cos\psi - R\cos\phi + i(\sin\psi - R\sin\phi)} = \frac{A + iB}{C + iD}.$$

But

$$\mathfrak{R}\left(\frac{A + iB}{C + iD}\right) = \mathfrak{R}\left(\frac{(A + iB)(C - iD)}{C^2 + D^2}\right) = \frac{AC + BD}{C^2 + D^2},$$

and hence

$$\Re\left(\frac{e^{\psi i}+z}{e^{\psi i}-z}\right)=$$

$$\frac{(\cos\psi+R\cos\phi)(\cos\psi-R\cos\phi)+(\sin\psi+R\sin\phi)(\sin\psi-R\sin\phi)}{(\cos\psi-R\cos\phi)^2+(\sin\psi-R\sin\phi)^2}$$

$$=\frac{1-R^2}{1+R^2-2R\cos(\phi-\psi)}.$$

From this, it follows immediately that

(7) $$u(R,\phi)=\Re(F(z)).$$

For the moment, let

$$ze^{-\psi i}=q\,;$$

then

$$F(z)=\frac{1}{2\pi}\int_0^{2\pi}f(\psi)\frac{1+q}{1-q}\,d\psi.$$

Now the expansion

$$\frac{1}{1-q}=1+q+q^2+\ldots=\sum_{n=0}^{\infty}q^n$$

is a geometric progression which converges for any complex $q$ such that $|q|<1$; this series converges absolutely for $|q|<1$ and uniformly in the interior of any closed region lying inside the circle $|q|=1$. Hence it follows also that for $|q|<1$,

$$\frac{1+q}{1-q}=(1+q)(1+q+q^2+\ldots)=1+2\sum_{n=1}^{\infty}q^n,$$

and hence that for $|z|=R<1$,

$$F(z)=\frac{1}{2\pi}\int_0^{2\pi}f(\psi)\left\{1+2\sum_{n=1}^{\infty}z^ne^{-n\psi i}\right\}d\psi.$$

Since the series in the integrand, for every $z$ such that $|z|<1$, converges uniformly in $\psi$, we can integrate the series termwise, which gives

(8) $$F(z)=\frac{1}{2\pi}\int_0^{2\pi}f\,d\psi+\frac{1}{\pi}\sum_{n=1}^{\infty}z^n\int_0^{2\pi}fe^{-n\psi i}d\psi.$$

This may also be written (on account of the identity $z = Re^{\phi i}$)

$$(8^*)\qquad F(z) = \frac{1}{2\pi}\int_0^{2\pi} f d\psi + \frac{1}{\pi}\sum_{n=1}^{\infty} R^n \int_0^{2\pi} f e^{n(\phi-\psi)i} d\psi.$$

By taking the real part of this, we find

$$u = \Re\,(F(z)) = \frac{1}{2\pi}\int_0^{2\pi} f d\psi + \frac{1}{\pi}\sum_{n=1}^{\infty} R^n \int_0^{2\pi} f \cos n(\phi - \psi) d\psi;$$

and by using the identity

$$\cos n(\phi - \psi) = \cos n\phi \cos n\psi + \sin n\phi \sin n\psi,$$

we get the desired expansion (see also Chapter VI, Art. 2)

$$(9)\qquad u(R, \phi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} R^n (a_n \cos n\phi + b_n \sin n\phi),$$

where the constants $a_n$ and $b_n$ have the values

$$(10)\qquad \begin{cases} a_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(\psi) \cos n\psi d\psi & (n = 0, 1, 2, \ldots.), \\ b_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(\psi) \sin n\psi d\psi & (n = 1, 2, 3, \ldots.). \end{cases}$$

These constants are known as the Fourier coefficients of the function $f(\psi)$.

The above expansion is valid for $R < 1$, or for the entire *interior of the unit circle.* Since the series, obtained by termwise differentiation with respect to $R$ or $\phi$ arbitrarily many times, likewise converge uniformly in any closed region lying in the interior of the unit circle (from the theory of power series), it is known that such series represent the derivatives of $u$.

From the preceding results, we get the following theorem, which is the analogue of Cauchy's theorem concerning the expansion in a power series of an analytic function in its region of regularity: *If $u$ is a potential function regular in any region $T$*

*and if $O$ is any point of $T$* (which may be taken as the origin without loss of generality), *then $u$ can be expanded in the form* (9). *The series converges in the interior of the largest circle about $O$ which contains only points of $T$ in its interior; it converges uniformly in any closed region lying in the interior of this maximum circle. It can be differentiated term-wise arbitrarily often in this circle.*

To prove this, it is only necessary to let $P$ be a point inside this maximum circle, and to take a concentric circle slightly smaller which still contains $P$, and apply the Poisson integral to this circle, which leads to the required development.

We note also the following fact: *the expansion of $u$ in the form* (9) *is unique*, or possible in only one way. For if there were a second such expansion,

$$u = \frac{a'_0}{2} + \Sigma R^n(a'_n \cos n\phi + b'_n \sin n\phi),$$

then it would follow that the series

$$\frac{a_0 - a'_0}{2} + \Sigma R^n ((a_n - a'_n) \cos n\phi + (b_n - b'_n) \sin n\phi) = 0$$

in some region about $O$ identically in $R$ and $\phi$. Then, for an arbitrary but fixed value of $\phi$, this is a power series in $R$ which vanishes identically for some interval about $O$. Hence the coefficient of each power of $R$ must vanish, so that

$$a_0 - a'_0 = 0$$

$$(a_n - a'_n) \cos n\phi + (b_n - b'_n) \sin n\phi = 0, \ (n = 1, 2, 3, \ldots).$$

Since these equations hold identically in $\phi$, we get the desired conclusion that $a_n = a'_n$, $b_n = b'_n$.

## Art. 3. Expansion on the Circumference of the Circle

It may happen that the series (9) converges for $R = 1$ and a definite value of $\phi$, that is, that the series

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\phi+b_n\sin n\phi) \tag{11}$$

with the coefficients (10) converges for a definite value of $\phi$. The series (11) is known as a *Fourier series*, after the mathematician and physicist Fourier, who first used such series in the theory of heat conduction. In case the Fourier series converges, then the expansion (9), on passing to the limit, gives an expansion valid on the boundary circle,

$$\lim u(R,\phi)=f(\phi)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\phi+b_n\sin n\phi). \tag{12}$$

To obtain this, use is made of the theorem that

$$\lim_{R\to 1}\sum_{n=1}^{\infty}R^n(a_n\cos n\phi+b_n\sin n\phi)=\sum(a_n\cos n\phi+l \tag{13}$$

or that the process of passing to the limit as $R\to 1$ may be interchanged with the $\Sigma$ sign. The justification for this equation is Abel's theorem in the theory of infinite series, which we will assume here. Let the series $\sum_{n=1}^{\infty}c_nR^n$ converge in the open interval $-1<R<1$. Then it converges uniformly in any smaller interval $-k\leq R\leq k$, where $0<k<1$, and represents a continuous function in this interval, say $G(R)$. Now if the series $\sum_{n=1}^{\infty}c_n$ converges, then according to Abel the point $R=1$ can also be brought into the region of uniform convergence, so that the series $\Sigma c_nR^n$ converges uniformly for $-k\leq R\leq 1$. Its sum $G(R)$ is therefore also continuous for $R=1$ (of course for approach from the left), or $G(R)\to G(1)$ as $R\to 1-0$. But this result is equivalent to the equation (13).

If the series (12) converges for every value of $\phi (0\leq\phi\leq 2\pi)$, then (12) is valid for all values of $\phi$, and $f(\phi)$ is developable in a Fourier series on the entire circumference of the circle. We

sec therefore that: at every point of the circumference at which the Fourier series with coefficients (10) converges, it actually converges to the function $f(\phi)$ (which has been assumed continuous). We also note that: if the function $f(\phi)$ is developable in a uniformly convergent Fourier series in the interval $0 \leq \phi \leq 2\pi$, then the coefficients $a_n$ and $b_n$ are certainly given by the equations (10). This is proved if we multiply (12) by $\cos n\phi$ or $\sin n\phi$ $(n = 0, 1, 2, \ldots.)$ and integrate termwise, remembering the orthogonality of the trigonometric functions (see Chapter IV, Art. 6). The expansion in a Fourier series is therefore unique.

## Art. 4. Expansion of Arbitrary Functions in Fourier Series. Bessel's and Schwarz's Inequalities

We were led to Fourier series in studying the boundary value problem for the circle, but will now study them independently of this derivation.

Let *the function $f(\phi)$ be integrable and of integrable square* in the interval $0 \leq \phi \leq 2\pi$.[1]

The Fourier coefficients of $f$ are

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f d\phi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f \cos n\phi d\phi, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f \sin n\phi d\phi, \quad n = 1, 2, \ldots$$

We form from these the $n$th "partial sum,"

$$s_n = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos k\phi + b_k \sin k\phi). \tag{14}$$

We will first prove that the sum of the squares of the Fourier coefficients

$$\frac{a_0^2}{4} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2) \tag{15}$$

[1]Continuity of $f$ is not assumed.

form a convergent series. By using the orthogonality properties of the trigonometric functions, we find

$$\int_0^{2\pi} [f(\phi) - s_n(\phi)]^2 d\phi = \int_0^{2\pi} f^2 d\phi - 2\int_0^{2\pi} f s_n d\phi + \int_0^{2\pi} s_n^2 d\phi$$

$$= \int f^2 d\phi - 2\left\{\frac{\pi a_0^2}{2} + \pi \sum_{k=1}^{n} (a_k^2 + b_k^2)\right\} + \frac{\pi a_0^2}{2} + \sum_{k=1}^{n} \pi(a_k^2 + b_k^2)$$

$$= \int_0^{2\pi} f^2 d\phi - \pi\left[\frac{a_0^2}{2} + \sum_{k=1}^{n} (a_k^2 + b_k^2)\right].$$

Since the first member of this identity is non-negative, we find that this gives

$$\frac{a_0^2}{2} + \sum_{k=1}^{n} (a_k^2 + b_k^2) \leq \frac{1}{\pi} \int_0^{2\pi} f^2 d\phi, \tag{16}$$

which is known as *Bessel's inequality*. Moreover, since this holds for all values of $n$, it is evident further that the sum of squares of the Fourier coefficients form a convergent series and that

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2) \leq \frac{1}{\pi} \int_0^{2\pi} f^2 d\phi. \tag{16*}$$

From this convergence it follows that

$$\lim_{n\to\infty} a_n = 0, \lim_{n\to\infty} b_n = 0, \tag{17}$$

so that the Fourier coefficients of a function integrable with integrable square converge to zero.

Since only the orthogonality properties of the trigonometric functions were used in the above proof, we can prove in the same manner that: If the functions $\phi_k(x)$, $(k = 1, 2, 3, \ldots)$, form a normal orthogonal system for the interval $a \leq x \leq b$ and if the function $f(x)$ is integrable with integrable square in this interval, then its Fourier coefficients $c_k = \int_a^b f(x)\phi_k(x)dx$ satisfy the Bessel inequality

$$\sum_{k=1}^{n} c_k^2 \leq \int_a^b f^2 dx \tag{16**}$$

for every $n$, and hence also for $n = \infty$. (The factor $\frac{1}{\pi}$ on the right side of (16) and (16*) occurs because the trigonometric functions are orthogonal but not normalized.) From (16**) it follows that $\sum c_k^2$ converges and $\lim\limits_{k} c_k = 0$.

It can be shown, as part of the general theory of orthogonal functions, that the equality sign holds instead of the inequality in (16*). This equation then becomes the "Parseval formula", or the "completeness condition." However, we cannot go any further into the study of "complete" orthogonal systems, to which the trigonometric functions belong.

From (16**) using $n = 1$, we find $c_1^2 = \left[\int_a^b f\phi_1 dx\right]^2 \leqq \int_a^b f^2 dx$; the function $\phi_1(x)$ is of course normalized, i.e., it satisfies the condition $\int_a^b \phi_1^2 dx = 1$. If $g(x)$ is an arbitrary integrable function with integrable square, then the function $\phi_1(x) = \dfrac{g(x)}{\sqrt{\int_a^b g^2 dx}}$ is normalized. Hence it follows that

$$\frac{\left[\int_a^b fg dx\right]^2}{\int_a^b g^2 dx} \leq \int_a^b f^2 dx$$

or

$$\left[\int_a^b fg dx\right]^2 \leq \int_a^b f^2 dx \int_a^b g^2 dx. \tag{16***}$$

This is the *Schwarz inequality*. It holds for any two functions $f$ and $g$ which are integrable with integrable square on the interval $a \leq x \leq b$. We have here derived it as a special case of Bessel's inequality; it can easily be derived directly, also.

Our problem is now to determine conditions which are sufficient to ensure that a function will be represented by its Fourier series. This problem is one of the most interesting and important of mathematical physics. There are different methods of finding such sufficient conditions. We will consider here a method which is essentially due to Dirichlet. For this purpose, we will first find a simple expression for the partial sum (14). This sum is

$$s_n(\phi) = \frac{1}{2\pi}\int f(\psi)d\psi + \frac{1}{\pi}\sum_{k=1}^{n}\left\{\cos k\phi \int f(\psi)\cos k\psi d\psi + \sin k\phi \int f(\psi)\sin k\psi d\psi\right\}$$

$$= \frac{1}{2\pi}\int f(\psi)\left\{1 + 2\sum_{k=1}^{n}\cos k(\psi - \phi)\right\}d\psi.$$

The interval of integration may be any interval of total length $2\pi$, since we will assume that $f(\phi)$ has the period $2\pi$.

By adding the equations

$$2\sin\frac{\alpha}{2}\cos k\alpha = \sin(2k+1)\frac{\alpha}{2} - \sin(2k-1)\frac{\alpha}{2},$$

we get the elementary identity

$$1 + 2\sum_{k=1}^{n}\cos k\alpha = \frac{\sin(2n+1)\dfrac{\alpha}{2}}{\sin\dfrac{\alpha}{2}}. \tag{18}$$

Hence

$$s_n(\phi) = \frac{1}{2\pi}\int f(\psi)\frac{\sin(2n+1)\dfrac{\psi-\phi}{2}}{\sin\dfrac{\psi-\phi}{2}}d\psi,$$

or, if we introduce the new integration variable $\dfrac{\psi - \phi}{2} = \theta$, this becomes

$$s_n(\phi) = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\phi + 2\theta)\,\frac{\sin(2n+1)\theta}{\sin\theta}\,d\theta. \tag{19}$$

In the last integral, the integration interval must have the length $\pi$, and this has been chosen as the interval $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

This formula was obtained by Dirichlet, who derived sufficient conditions from it (that $f$ be represented by the Fourier series). The continuity of $f$ is not sufficient. For, it was first proved by du Bois Reymond in an article in the "Abhandlungen der Bayrischen Akademie der Wissenschaften," Band 12 (1876), that the Fourier series for a continuous function may not converge. On the other hand, continuity is not necessary, as we shall soon see. The conditions under which Dirichlet established the convergence, or the existence of $\lim\limits_{n\to\infty} s_n(\phi)$, require that $f$ be continuous and monotone in the interval $0<\phi<2\pi$ or that this interval may be divided into a finite number of sub-intervals in which $f$ is continuous and monotone.

We will again modify the formula for $s_n(\phi)$, and then derive other sufficient conditions. We separate the integral (19) into the sum of two integrals $\int_0^{\frac{\pi}{2}} \ldots\ldots + \int_{-\frac{\pi}{2}}^{0} \ldots\ldots$, make the change of variables $-\theta = \theta'$ in the second integral, and then write this new integration variable as $\theta$ instead of $\theta'$, obtaining

$$s_n(\phi) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} [f(\phi + 2\theta) + f(\phi - 2\theta)]\,\frac{\sin(2n+1)\theta}{\sin\theta}\,d\theta.$$

Since $s_n(\phi) = 1$ for $f(\phi) = 1$, it follows that

$$1 = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)\theta}{\sin\theta}\,d\theta.$$

Multiplying this by $f(\phi)$ and subtracting $s_n(\phi)$, we find

$$(20) \qquad f(\phi) - s_n(\phi) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \left\{2f(\phi) - f(\phi + 2\theta) - f(\phi - 2\theta)\right\} \frac{\sin(2n+1)\theta}{\sin\theta}\,d\theta,$$

or introducing the abbreviations $F(\phi)$ for the right member of (20) and $\left\{2f(\phi) - f(\phi + 2\theta) - f(\phi - 2\theta)\right\} = g(\phi, \theta)$,

$$(20^*) \qquad f(\phi) - s_n(\phi) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \frac{g(\phi, \theta)}{\sin\theta} \sin(2n+1)\theta d\theta = F(\phi).$$

The difficulty in the discussion of $F(\phi)$ arises from the fact that the denominator $\sin\theta$ vanishes at the point $\theta = 0$. We will now assume that

$$(21) \qquad \left|\frac{g(\phi, \theta)}{\sin\theta}\right| \leqq M$$

holds for some interval $|\theta| \leqq c$, i.e. for some interval about the point $\theta = 0$, while we will consider $\phi$ as fixed. Here $c$ and $M$ are positive constants. We now write the integral in the form

$$F(\phi) = \frac{1}{\pi}\int_0^{\delta} \cdots\cdot + \frac{1}{\pi}\int_{\delta}^{\frac{\pi}{2}} \cdots\cdot$$

where $\delta$ satisfies the condition $0 < \delta \leqq c$. Then, since $|\sin(2n+1)\theta| \leqq 1$,

$$\left|\frac{1}{\pi}\int_0^{\delta} \frac{g(\phi, \theta)}{\sin\theta} \sin(2n+1)\theta d\theta\right| \leqq \frac{M}{\pi}\delta.$$

For any given small positive quantity $\epsilon$, we can choose $\delta$ small enough so that $\frac{M\delta}{\pi} < \frac{\epsilon}{2}$. Having selected $\delta$ in this manner, we consider the integral

$$\frac{1}{\pi}\int_{\delta}^{\frac{\pi}{2}} \frac{g(\phi, \theta)}{\sin\theta} \sin(2n+1)\theta d\theta.$$

This integral causes no difficulty, because the denominator of the integrand does not vanish in the interval of integration. Let $h(\theta)$ be that function which is identical with $\dfrac{g(\phi, \theta)}{\sin\theta}$ in the interval $\delta \leq \theta \leq \dfrac{\pi}{2}$ and vanishes identically in the remainder of the interval $0 \leq \theta < 2\pi$, so that $h(\theta)$ is a function of integrable square. The integral becomes

$$\frac{1}{\pi}\int_0^{2\pi} h(\theta)\sin(2n+1)\theta d\theta.$$

This, however, is the Fourier coefficient $b_{2n+1}$ of $h(\theta)$, and therefore approaches zero with increasing $n$. The absolute value of this integral therefore becomes less than $\dfrac{\epsilon}{2}$ when $n$ is sufficiently large. Therefore it follows that

$$|f(\phi) - s_n(\phi)| < \epsilon \tag{22}$$

for sufficiently large $n$, establishing the convergence.

The condition (21) is therefore a sufficient condition to ensure the convergence of the Fourier series to $f(\phi)$ at the point $\phi$. *If* (21) *holds uniformly in* $\phi$ *and* $\theta$, that is, if there exist two constants $c$ and $M$ independent of $\phi$ and $\theta$ such that (21) holds for $0 \leq \phi \leq 2\pi$ and $|\theta| \leq c$, *then* $f(\phi)$ *can be expanded in a uniformly convergent Fourier series in the interval* $0 \leq \phi \leq 2\pi$.

*The condition* (21) *certainly holds uniformly if the periodic function* $f(\phi)$ *is continuous with continuous first derivative in the interval* $0 \leq \phi \leq 2\pi$. For

$$f(\phi + 2\theta) - f(\phi) = 2\theta f'(\xi) \quad (\phi < \xi < \phi + 2\theta)$$

$$\left|\frac{f(\phi + 2\theta) - f(\phi)}{\sin\theta}\right| = \left|\frac{2\theta f'(\xi)}{\sin\theta}\right|.$$

But $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta} = 1$, and it is easily seen that $\left|\dfrac{\theta}{\sin\theta}\right| < 2$ holds

in $0 \leq \theta \leq \frac{\pi}{2}$. Let $C$ be the maximum value of the function $|f'(\phi)|$ for $0 \leq \phi \leq 2\pi$ (which exists because $f'$ is continuous), then

$$\left|\frac{f(\phi+2\theta)-f(\phi)}{\sin\theta}\right| \leq 4C, \text{ and similarly } \left|\frac{f(\phi-2\theta)-f(\phi)}{\sin\theta}\right| \leq 4C,$$

hence

$$\left|\frac{g(\phi,\theta)}{\sin\theta}\right| \leq 8C$$

so that (21) is satisfied with $M = 8C$ and $c = \frac{\pi}{2}$.

Also, (21) is uniformly satisfied if the hypothesis is merely that $f(\phi)$ is piece-wise continuous with piece-wise continuous first derivative, and at discontinuities satisfies the condition

$$f(\phi) = \tfrac{1}{2}\{f(\phi-0)+f(\phi+0)\}. \tag{23}$$

For at the discontinuities,

$$f(\phi+2\theta)-f(\phi+0) = 2\theta f'(\xi),\ f(\phi-2\theta)-f(\phi-0) = 2\theta f'(\xi_1),$$

from which it follows that

$$\left|\frac{f(\phi+2\theta)+f(\phi-2\theta)-f(\phi+0)-(f\phi-0)}{\sin\theta}\right| \leq 8C = M,$$

so that it is seen that (21) is satisfied on account of (23).

*The Fourier series for a function which is piece-wise continuous with its first derivative converges to the function itself at the points of continuity, and converges to the arithmetic mean* (23) *at the points of discontinuity.*

The condition (21) may be replaced by the less stringent condition

$$\left|\frac{g(\phi,\theta)}{\sin\theta}\right| \leq \frac{M}{\theta^{1-a}}, \tag{21*}$$

where $a$ is a positive constant (in (21), $a = 1$). This condition

requires that $\theta^{1-a}\dfrac{g(\phi, \theta)}{\sin\theta}$ remain bounded for $\theta \to 0$; on the other hand, $\dfrac{\quad}{\sin\theta}$ need not be bounded. Using this condition, we find

$$\left|\frac{1}{\pi}\int_0^\delta \frac{g(\phi, \theta)}{\sin\theta}\sin(2n+1)\theta d\theta\right| \leq \frac{M}{\pi}\int_0^\delta \frac{1}{\theta^{1-a}}d\theta = \frac{M}{\pi a}\delta^a,$$

which can again be made less than $\dfrac{\epsilon}{2}$ by choosing $\delta$ sufficiently small. The remainder of the proof is as before.

The condition (21*) is certainly fulfilled if the function $f(\phi)$ satisfies at $\phi$ a "*Hölder condition*," i.e. if there exist three constants $c$, $C$, $a$ such that

$$|f(\phi+\theta) - f(\phi)| \leq C\theta^a \text{ for } |\theta| \leq c. \tag{24}$$

(A function which satisfies a Hölder condition is evidently continuous, but is not necessarily differentiable.) For, by (24),

$$\left|\frac{f(\phi \pm 2\theta) - f(\phi)}{\sin\theta}\right| = \left|\frac{f(\phi \pm 2\theta) - f(\phi)}{\theta}\,\frac{\theta}{\sin\theta}\right| < \frac{2C(2\theta)^a}{\theta} = \frac{2^{1+a}C}{\theta^{1-a}}$$

and (21*) follows easily.

If $f(\phi)$ satisfies a Hölder condition uniformly in $0 \leq \phi \leq 2\pi$ (i.e., if the constants $c$, $C$, $a$ are independent of $\phi$), then the Fourier series converges uniformly to $f(\phi)$ in the interval. Moreover, it converges uniformly to $f(\phi)$, if $f(\phi)$ is piece-wise continuous and satisfies (24) in each interval where it is continuous, and satisfies (23) at the discontinuities.

## Art. 5. Expansion in a Circular Ring

Following the expansion of a potential function inside a circle, obtained in Article 2, we will obtain an expansion valid

in a circular ring-shaped region. This is the analogue of the Laurent series in the theory of functions of a complex variable.

Let the potential $u$ be regular in the region $T$ bounded by the circles $C_1$ and $C_2$, of radii $l_1$ and $l_2$ respectively ($l_1 > l_2$), about the origin $O$ as centre. Let $P$ be a point of $T$. Let $C'_1$ and $C'_2$ be circles about the same centre $O$, of radii $R_1$ and $R_2$ ($R_1 > R_2$), where $R_1$ is slightly less than $l_1$ and $R_2$ is slightly greater than $l_2$, so that $P$ is between $C'_1$ and $C'_2$. Then we can apply Equation (38) of Chapter III to the region $T^*$ between these circles, to obtain

$$(25) \qquad u = \frac{1}{2\pi}\int_{C'_1}\left\{\log\frac{1}{r}\frac{\partial u}{\partial n} - u\frac{\partial\left(\log\frac{1}{r}\right)}{\partial n}\right\}ds + \frac{1}{2\pi}\int_{C'_2}\left\{\log\frac{1}{r}\frac{\partial u}{\partial n} - u\frac{\partial\left(\log\frac{1}{r}\right)}{\partial n}\right\}ds$$

$$= u_1 + u_2.$$

We now consider the functions $u_1$ and $u_2$ separately. It is evident that each of them is the potential due to a linear distribution and a double layer. They are therefore regular for all points not lying on the circles $C'_1$ and $C'_2$. In particular, the potential $u_1$ is regular everywhere inside $C'_1$, and can be expanded there (and hence at $P$) in the form

$$(26) \qquad u_1 = \frac{a_0}{2} + \sum_{n=1}^{\infty} R^n(a_n \cos n\phi + b_n \sin n\phi).$$

The potential $u_2$ is certainly regular for all finite points outside $C'_2$. The portion $-\dfrac{1}{2\pi}\displaystyle\int_{C'_2} u\frac{\partial\left(\log\frac{1}{r}\right)}{\partial n}ds$, being the potential of a double distribution, is also regular at infinity. On the other

hand, the term $\frac{1}{2\pi}\int_{C'_2} \log \frac{1}{r}\frac{\partial u}{\partial n}\, ds$ behaves at infinity like a potential of mass $M = \frac{1}{2\pi}\int_{C'_2} \frac{\partial u}{\partial n}\, ds.$ Hence this is true also for the function $u_2$ itself, so that

$$u_2 = M \log \frac{1}{R} + v, \tag{27}$$

where the potential $v$ is regular everywhere outside $C'_2$ and vanishes at infinity.[2] Moreover, the value of the constant $M$ is independent of the radius $R_2$ of the circle $C'_2$; for the integral $\int_C \frac{\partial u}{\partial n}\, ds$ must, of course, have the same value for any circle between $C_1$ and $C_2$.

The potential $v$ is transformed by an inversion $R' = 1/R$, $\phi' = \phi$, into a potential which is regular in the neighbourhood of the origin of the $x'$, $y'$-plane and vanishes there. Hence this transformed function has an expansion of the form

$$v\left(\frac{1}{R'}, \phi'\right) = \sum_{n=1}^{\infty} R'^n (a_{-n} \cos n\phi' + b_{-n} \sin n\phi').$$

This expansion is valid for $R' < 1/R_2$. Hence we have

$$v = \sum_{n=1}^{\infty} R^{-n}(a_{-n} \cos n\phi + b_{-n} \sin n\phi) \tag{28}$$

for $R > R_2$; hence this is valid outside $C'_2$. The series for $u_1$ and $v$ have the common region of convergence $R_2 < R < R_1$. Thus throughout this region, and hence at the point $P$, we have the expansion

$$u = \tfrac{1}{2}(a_0 + a_0' \log R) + \sum_{n=1}^{\infty} \left\{(a_n R^n + a_{-n}R^{-n}) \cos n\phi + (b_n R^n + b_{-n}R^{-n}) \sin n\phi\right\}, \tag{29}$$

[2] See equation (39) and the end of Art. 7 in Chapter 3. We have $c = 0$.

where $M = -\frac{a_0}{2}$. This is the desired expansion. Since $P$ is an arbitrary point of the region $T$, this expansion is valid throughout the entire region. Hence: *A potential which is regular in a ring-shaped region about the origin as centre may be expanded in a series of the form* (29). *The series is uniformly convergent in any closed region lying entirely interior to the ring, and may be differentiated there arbitrarily often.*

The uniform convergence and term-wise differentiability follow from that of the corresponding series for the functions $u_1$ and $v$. The coefficients of the expansion may be found in the following manner. First, for any $R$ in the range $l_2 < R < l_1$,

$$a_0 + a_0' \log R = \frac{1}{\pi}\int_0^{2\pi} u\,(R,\phi)\,d\phi,$$

$$\left.\begin{aligned} a_nR^n + a_{-n}R^{-n} &= \frac{1}{\pi}\int_0^{2\pi} u\,(R,\phi)\cos n\phi\,d\phi,\\ b_nR^n + b_{-n}R^{-n} &= \frac{1}{\pi}\int_0^{2\pi} u(R,\phi)\sin n\phi\,d\phi,\end{aligned}\right\} n = 1, 2, \ldots$$

Now selecting two values $R_1$ and $R_2$ for $R$, satisfying the inequalities $l_2 < R_2 < R_1 < l_1$, we find that

$$a_0 = \frac{a_{20}\log R_1 - a_{10}\log R_2}{\log R_1 - \log R_2},\quad a'_0 = \frac{a_{10} - a_{20}}{\log R_1 - \log R_2},$$

$$(30)\quad a_n = \frac{a_{1n}R_1^n - a_{2n}R_2^n}{R_1^{2n} - R_2^{2n}},\quad a_{-n} = \frac{a_{1n}R_1^{-n} - a_{2n}R_2^{-n}}{R_1^{-2n} - R_2^{-2n}},$$

$$b_n = \frac{b_{1n}R_1^n - b_{2n}R_2^n}{R_1^{2n} - R_2^{2n}},\quad b_{-n} = \frac{b_{1n}R_1^{-n} - b_{2n}R_2^{-n}}{R_1^{-2n} - R_2^{-2n}},$$

where

$$(31)\quad \left.\begin{aligned} a_{kn} &= \frac{1}{\pi}\int_0^{2\pi} u(R_k,\phi)\cos n\phi\,d\phi\\ b_{kn} &= \frac{1}{\pi}\int_0^{2\pi} u(R_k,\phi)\sin n\phi\,d\phi\end{aligned}\right\}\begin{aligned} k &= 1, 2\\ n &= 0, 1, 2, \ldots\ldots\end{aligned}$$

From this determination of the coefficients, it follows that *the expansion* (29) *is unique.* For if there were two expansions possible, then by subtraction it would follow that ($c_0, c_0', \ldots d_{-n}$ being constants)

$$0 = \tfrac{1}{2}(c_0 + c_0' \log R) + \sum_{n=1}^{\infty} \{(c_n R^n + c_{-n} R^{-n}) \cos n\phi + (d_n R^n + d_{-n} R^{-n}) \sin n\phi\}$$

holds identically in the ring-shaped region $T$. First, the constants $c_{kn}$ and $d_{kn}$ corresponding to $a_{kn}$, $b_{kn}$, given by formulas (31) with $u \equiv 0$, are all zero; hence the coefficients $c_0$, $c'_0$, $c_n$, $c_{-n}$, $d_n$, $d_{-n}$ are all zero.

Special cases of the above expansion are obtained when $l_2 = 0$ or $l_1 = \infty$. The case $l_2 = 0$ is especially important, giving the expansion of a potential in the neighbourhood of an *isolated*[3] *singularity.* The expansion (29) is then valid everywhere, except at the origin (centre), inside the circle of radius $l_1$.

We can now prove the following theorem (essentially due to H. A. Schwarz): *If the logarithmic potential u is regular in the neighbourhood of a finite point O (with the possible exception of the point O itself) and remains bounded on approaching O, then u is also regular at O.* For we may take $O$ as the origin and use the expansion (29) in its neighbourhood, $0 < R < l_1$. In the determination of the coefficients, we may take $R_2$ as small as we please. By the hypothesis, the coefficients $a_{kn}$ and $b_{kn}$ remain bounded as $R_2 \to 0$. Hence it follows from (30) that the coefficients $a'_0$, $a_{-n}$, $b_{-n}$ may be made as small in absolute value as desired by taking $R_2$ sufficiently small. But since they are constants and do not depend on $R_2$ for their values, they are all zero. This was to be proved.

[3]A singular (that is, not regular) point $P$ of $u$ is called an "isolated" singularity if $u$ is regular at all points of a neighbourhood of $P$, except $P$ itself. The reader should give examples.

*The above theorem remains valid when $O$ is the point at infinity.* The proof is immediate, after using the inversion to change the point at infinity into the origin.

The hypothesis of boundedness in the region of infinity is certainly satisfied if $\lim\limits_{R\to} u$ exists. Hence the theorem may be put in the following special form: *If the potential $u$ is regular in the neighbourhood of infinity* (*except perhaps at infinity*), *and* $\lim\limits_{R\to\infty} u$ *exists, then $u$ is regular at infinity.* It follows, of course, that $R^2D_1u$ is bounded (see Chapter II, Art. 7).

## Art. 6. The Equipotentials are Analytic Curves

By using the expansion of a harmonic function in the interior of a circle, we will now obtain important properties of the equipotential curves. Let the potential $u$, regular in the neighbourhood of the point $O$, take on the value $u_0$ there; we will take $O$ as the origin, and investigate the equipotential line

$$u(x, y) = u_0 \tag{32}$$

(a constant) in the neighbourhood of $O$.

The function $u(x, y)$ is analytic (Chapter IV, Art. 1) at $O$; if at least one of the partial derivatives, say $\dfrac{\partial u}{\partial x}$, is not zero at $O$, then $O$ is a regular point of the curve. By the implicit function theorem, there exists one and only one analytic function $y = f(x)$, which is regular at $x = 0$, takes on the value 0 there, and satisfies $u = u_0$ identically in $x$.

In case both partial derivatives $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial u}{\partial y}$ vanish at $O$, then $O$ is a singular point of the equipotential (32), and is indeed a multiple point, as will be seen at once. Let $m$, $\geq 2$, be the order of the lowest derivative which does not vanish at $O$; then we have in the neighbourhood of $O$ the expansion

$$u = u_0 + \sum_{n=m}^{\infty} R^n(a_n \cos n\phi + b_n \sin n\phi),$$

since the coefficients of $R, R^2, \ldots\ldots, R^{m-1}$ all vanish, while $a_m$, $b_m$ cannot both be zero.

In order to solve (for $\phi$) the equation

$$(32^*) \qquad u - u_0 = \sum_{n=m}^{\infty} R^n(a_n \cos n\phi + b_n \sin n\phi) = 0,$$

or the equivalent equation .

$$(32^{**}) \quad H(R, \phi) = a_m \cos m\phi + b_m \sin m\phi + R(\ldots\ldots) = 0,$$

in the neighbourhood of the origin, we may proceed as follows. First find a pair of values $R = 0$, $\phi = \phi_0$ satisfying the equation; that is, $\phi_0$ must satisfy the equation

$$a_m \cos m\phi + b_m \sin m\phi = 0.$$

From one solution of this equation, all the others are obtained by the addition of $\dfrac{k\pi}{m}$ where $k$ is any integer. Hence the equation has exactly $m$ solutions in the range $0 \leqq \phi < \pi$, spaced at the equal angles $\dfrac{\pi}{m}$. The derivative

$$\frac{\partial H}{\partial \phi} = m(-a_m \sin m\phi + b_m \cos m\phi) + R(\ldots)$$

cannot vanish for $R = 0$, $\phi = \phi_0$. For this would mean that simultaneously

$$-a_m \sin m\phi_0 + b_m \cos m\phi_0 = 0,$$
$$b_m \sin m\phi_0 + a_m \cos m\phi_0 = 0,$$

so that $a_m^2 + b_m^2 = 0$; but $a_m$ and $b_m$ are not both zero. Hence there is one and only one analytic function $\phi = \phi(R)$ which is regular near $R = 0$ and takes the value $\phi_0$ there, and satisfies $H(R, \phi) = 0$ identically. Corresponding to the $m$ different values $\phi_0$, there are $m$ such analytic functions $\phi(R)$. The cor-

responding curves are identical with the curve $u = u_0$ in the neighbourhood of $O$. They cut each other in $O$ at the equal angles $\frac{\pi}{m}$. Hence we have proved that: *The curves $u = const.$ are analytic curves. Any singular point can only be a multiple point, in which a finite number of branches meet at equal angles.*

Other types of singularities, such as corners, cusps, end-points, and isolated points, cannot occur. For example, in the potential $u = x^2 - y^2$, the equipotential lines $u = 0$ are the straight lines $y = x$ and $y = -x$, which cut at the angles $\frac{\pi}{2}$.

### Art. 7. Harnack's Theorems

Let $D$ be a finite plane region bounded by $C$. We will prove the following theorem, due to A. Harnack: *If the series*

$$\sum_{n=1}^{\infty} u_n(x, y), \tag{33}$$

*with terms $u_n$ which are regular potential functions in $D$ and continuous in the closed region $D + C$, converges uniformly on the boundary $C$, then it converges uniformly in the region $D + C$ and represents a continuous potential function $u$ there. The series may be differentiated term-wise arbitrarily often in any region interior to $D$. The derived series converge uniformly in $D$ and represent the corresponding derivatives of $u$.*

The proof rests on the theorems about the positions of maxima and minima, and on the Poisson integral. On account of the uniform convergence, for any positive $\epsilon$ there is an $n$, such that for any point $A$ on $C$ and any integer $p$,

$$|u_{n+1}(A) + u_{n+2}(A) + \ldots + u_{n+p}(A)| < \epsilon.$$

The $n$ is independent of $A$, and dependent only on $\epsilon$. Now the finite sum

$$u_{n+1}(x, y) + u_{n+2}(x, y) + \ldots + u_{n+p}(x, y)$$

represents a regular harmonic function in $D$, continuous in $D + C$. Its values for all points of $C$ lie in the range between $-\epsilon$ and $+\epsilon$. Hence in the entire region $D + C$ (remember the position of the extrema)

$$|u_{n+1}(x, y) + u_{n+2}(x, y) + \ldots + u_{n+p}(x, y)| < \epsilon,$$

from which the uniform convergence of (33) follows. The function

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x, y) \tag{34}$$

is continuous in $D + C$, being the sum of a uniformly convergent series of continuous functions. In order to show that it is harmonic, we use the Poisson integral. Let $P$:$(x, y)$ be an arbitrary point of $D$. Take a circle $K$ which lies in $D$ and contains $P$ in its interior. Use the centre of the circle $K$ as origin, and let $l$ be its radius. Then by Poisson's integral, Equation (2),

$$u_n(x, y) = \frac{1}{2\pi}\int_0^{2\pi} u_n(\psi)\,\frac{l^2 - R^2}{r^2}\,d\psi, \quad n = 1, 2, \ldots, \tag{35}$$

where, of course, $u_n(\psi)$ means the value of $u_n(x, y)$ on the circle $K$. Hence

$$u(x, y) = \sum_{n=1}^{\infty} \frac{1}{2\pi}\int_0^{2\pi} u_n(\psi)\,\frac{l^2 - R^2}{r^2}\,d\psi.$$

But since $\sum\limits_{n=1}^{\circ} u_n$ converges uniformly in $D + C$, and hence on the circle $K$, this is also true for $\Sigma u_n(\psi)\dfrac{l^2 - R^2}{r^2}$; hence we may integrate term-wise, which gives

$$u(x, y) = \frac{1}{2\pi}\int_0^{2\pi}\left\{\sum_{n=1}^{\infty} u_n(\psi)\right\}\frac{l^2 - R^2}{r^2}\,d\psi,$$

or

(36) $$u(x, y) = \frac{1}{2\pi}\int_0^{2\pi} u(\psi)\,\frac{l^2 - R^2}{r^2}\,d\psi,$$

where $u(\psi)$ is the value of $u(x, y)$ on the circle $K$. Thus the function $u$ is represented in the interior of $K$ by a Poisson integral over the continuous boundary values $u(\psi)$, so that $u$ is a harmonic function there and hence is harmonic in the neighbourhood of $P$. But since $P$ is an arbitrary point in $D$, the function $u$ is harmonic throughout $D$.

In a similar manner, it can be shown that: *a series of regular potential functions which is uniformly convergent in every sub-region of $D$ represents a potential regular in $D$.*

As for the partial derivatives of $u(x, y)$, we have for all points inside the circle $K$

$$\frac{\partial u}{\partial x} = \frac{1}{2\pi}\int_0^{2\pi} u(\psi)\,\frac{\partial}{\partial x}\left(\frac{l^2 - R^2}{r^2}\right)d\psi,$$

and for $n = 1, 2, 3, \ldots$

$$\frac{\partial u_n}{\partial x} = \frac{1}{2\pi}\int_0^{2\pi} u_n(\psi)\,\frac{\partial}{\partial x}\left(\frac{l^2 - R^2}{r^2}\right)d\psi\ ;$$

so that, again by term-wise integration,

(37) $$\sum_{n=1}^{\infty} \frac{\partial u_n}{\partial x} = \frac{1}{2\pi}\int_0^{2\pi}\left\{\Sigma u_n(\psi)\right\}\frac{\partial}{\partial x}\left(\frac{l^2 - R^2}{r^2}\right)d\psi = \frac{\partial u}{\partial x}.$$

This equation is valid for all points $P$ in $D$.

Since the series $\dfrac{\partial}{\partial x}\left(\dfrac{l^2 - R^2}{r^2}\right)\sum_{n=1}^{\infty} u_n(\psi)$ converges for all points $\psi$ on the circumference of the circle $K$ and all points $(x, y)$ of any sub-region of $K$, and converges uniformly, $\sum_{n=1}^{\infty}\dfrac{\partial u_n(x, y)}{\partial x}$ converges uniformly in any such region. Hence the uniform convergence follows in any circular region lying interior to $D$. Similarly it can be shown in general that

$$\frac{\partial^{p+q}u(x, y)}{\partial x^p \partial y^q} = \sum_{n=1}^{\infty} \frac{\partial^{p+q}u_n(x, y)}{\partial x^p \partial y^q} \tag{38}$$

is valid in the entire region $D$.

The series above converges uniformly in every circle interior to $D$. In order to show that the uniform convergence holds for any sub-region of $D$ (lying in its interior), we use the Heine-Borel theorem: If every point of a bounded closed region is an interior point of a circle, then the region is covered by a *finite* number of these circles. The convergence as stated above follows immediately from this. This completes the proof of Harnack's theorem.

A second theorem due to Harnack is: *If $u_1(x, y)$, $u_2(x, y)$ ........, is an infinite sequence of regular non-negative harmonic functions in $D$, and if the series $\Sigma u_n$ converges at a point $O$ of $D$, then it converges uniformly in any interior sub-region of $D$ and represents a regular harmonic function in $D$.*

To prove this, construct about $O$ as centre a circle $K$ of radius $l$, as large as possible but lying interior to the region $D$; then for all interior points of $K$,

$$u_n(x, y) = \frac{1}{2\pi}\int_0^{2\pi} u_n(\psi)\, \frac{l^2 - R^2}{r^2}\, d\psi.$$

Since both $u_n$ and $l^2 - R^2$ are non-negative, the value of each integral is increased when $r$ is decreased. The smallest value which $r$ can take is $l - R$. Hence

$$u_n(x, y) \leq \frac{1}{2\pi}\int_0^{2\pi} u_n(\psi)\frac{l^2 - R^2}{(l-R)^2}\, d\psi = \frac{1}{2\pi}\,\frac{l + R}{l - R}\int_0^{2\pi} u_n(\psi) d\psi,$$

so that

$$u_n(x, y) \leq \frac{l + R}{l - R} u_n(O) \tag{39}$$

from the mean value theorem. The inequalities (39) hold for

all interior points of $K$. From these inequalities, it is seen that the series converges for all such points $P$, and converges uniformly for any closed region lying entirely interior to the circle $K$. It represents therefore a regular potential in the interior of $K$. Now let $O'$ be a point inside $K$ but near its circumference, and describe about $O'$ a circle $K'$ as large as possible in $D$, in general extending outside of $K$. Then the series converges at $O'$ and hence inside the circle $K'$ and represents a regular potential function there. By continuing the function in this way, we can prove that the series converges and represents a potential function at any interior point $P$ of $D$. Moreover, the series converges uniformly in any circle lying inside $D$, and hence in any closed region interior to $D$ (from the Heine-Borel theorem).[4]

**Art. 8. Harmonic Continuation**

Analogous to the method of analytic continuation used in the theory of functions, we have for potentials or harmonic functions a method of "analytic continuation of a regular potential" or "harmonic continuation." This is defined as follows: A harmonic function $u$, defined and regular in a region $D$, is analytically continued when a function is defined which is regular and harmonic in a region $D'$, which extends outside $D$ but has a region $D^*$ in common with $D$, this function being identical with $u$ in the common region $D^*$.

*The analytic continuation of a harmonic function across a definite piece of boundary is possible in only one way.* For if $v$ and $w$ are regular potentials in $D'$ and are identical with $u$ in $D^*$, then $v - w$ is a regular potential in $D'$ and is identically zero in the sub-region $D^*$ of $D'$; from which it follows that

[4]Replacing in the above theorem the condition "non-negative" functions by "monotone" sequence (that is either $u_1 \geq u_2 \geq u_3 \ldots$ or $u_1 \leq u_2 \leq u_3 \ldots$ everywhere in $D$), one gets another Harnack Theore The reader should prove it.

$v - w$ must vanish identically throughout $D'$ (Chapter IV, Art. 1), so that $v$ and $w$ are identical.

The method, used in function theory, of analytic continuation by means of power series has its analogue here in expansions of the form (9) of Art. 2. The expansion is valid in the maximum circle, just as in the theory of functions.

On the other hand, we must investigate more carefully the process of "reflection" (well known in theory of functions). The fundamental theorem is: *If the potential $u$ is regular in a region $D$ which has a segment $AB$ of the $x$-axis as part of its boundary, and if $u$ is continuous in $D + AB$ and vanishes identically at the inner points of $AB$, then $u$ can be analytically continued into the reflection $D'$ of $D$ in the $x$-axis by assigning to the continued function at $P'$ the value $-u(P)$, where $P'$ is the reflection of $P$.* First, it is evident that the function

$$F = \begin{cases} u(P) \text{ at the point } P \text{ of } D \\ 0 \text{ at the inner points of } AB \\ -u(P) \text{ at the point } P' \text{ symmetric to } P \end{cases} \tag{40}$$

is continuous in the entire region $D + D'$, and in particular is continuous at the inner points of the segment $AB$. Moreover, it is a regular potential in $D'$ as well as in $D$. We will use the Poisson integral to prove that $F$ is also a regular potential at the points in the segment $AB$. Let $O$ be any inner point of the segment $AB$, and take a circle $K$ about $O$ of radius $l$, small enough to lie in $D + D'$. Consider the function

$$\Phi = \frac{1}{2\pi}\int_0^{2\pi} F\frac{l^2 - R^2}{r^2}\,d\psi. \tag{41}$$

This function is a regular harmonic function in the interior of $K$; it vanishes identically at points on the line-segment $AB$ since the symmetric elements of integration cancel in pairs. Also, in either half-circle it is identical with $F$, since $F$ and $\Phi$ take on the same boundary values in either half-circle and are

harmonic within it. But $F$ and $\Phi$ are also identical along the segment $AB$, so that $F \equiv \Phi$ everywhere inside $K$. Hence $F$ is a regular harmonic function inside $K$ and thus, in particular, at the point $O$ on $AB$. This completes the proof.

The above theorem may be generalized as follows: *If the potential $u$, regular in $D$, takes on the value $g(x)$ on $AB$ where $g$ is an analytic function, then $u$ can be analytically continued across $AB$.*

Because it is analytic, $g(x)$ can be expanded in a power series

$$g(x) = \sum_{n=0}^{\infty} c_n(x - x_0)^n$$

in the neighbourhood of any inner point $x_0$ of $AB$. It is known that this series also converges for complex values of its argument and represents an analytic function of the complex variable $z = x + iy$ in a circular region about $z = x_0$. This function $g(z) = \Sigma c_n(z - x_0)^n$ is therefore analytic in a circle extending to both sides of $AB$, and takes on the value $g(x)$ on the line $AB$. The real part $u_1$ of $g(z)$ is harmonic inside this circle and takes on the value $g(x)$ on $AB$. Hence $u$-$u_1$ is a regular potential defined in a certain neighbourhood above the line $AB$ and vanishing on this line, so that, by the preceding theorem, it may be analytically continued across $AB$. But since $u_1$ is already defined below the line $AB$, this gives an analytic continuation of $u$.

A still more general theorem is: *If $u$ is a regular potential in the region $D$, which has an analytic curve $AB$ without singular points as part of its boundary, and if $u$ takes on analytic values along $AB$, then $u$ can be analytically continued across $AB$.* Let $O$ be an arbitrary inner point of $AB$, and let the equation of the analytic arc $AB$ in the neighbourhood of $O$ be

$$x = \phi(t) = a_0 + a_1(t - t_0) + a_2(t - t_0)^2 + \dots$$
$$y = \psi(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)^2 + \dots,$$

with the point $O$ corresponding to the parameter value $t_0$. On $AB$ we have $u(x, y) = u(\phi, \psi)$, and this is, according to our assumption, an analytic function of $t$. Since $O$ is not a singular point, $a_1$ and $b_1$ cannot both be zero. Form the series

$$z = x + iy = \phi(t) + i\psi(t)$$

This series converges also for complex values of its variable, which may be designated by $Z$, and is an analytic function

$$z = f(Z) = a_0 + ib_0 + (a_1 + ib_1)(Z - t_0) + \ldots .$$

defined in some neighbourhood of $Z = t_0$. Since

$$\left.\frac{dz}{dZ}\right|_{Z = t_0} = a_1 + ib_1 \neq 0,$$

there exists a unique inverse function, $Z = F(z)$, which is a regular analytic function in some region about $z_0 = a_0 + ib_0$, takes on the value $t_0$ at $z_0$ and satisfies $z = f(Z)$ identically. This function maps the neighbourhood of $z = z_0$, in a one-to-one reversible manner, conformally on the neighbourhood of $Z = t_0$. By this map, the curve $AB$ in the neighbourhood of $O$ maps into a portion of the real axis of the $Z$-plane, containing the point $Z = t_0$. If we set $Z = X + iY$, the potential $u$ defined on one side of the curve $AB$ becomes a potential $U$ of $(X, Y)$[5], defined on one side of the axis of reals of the $Z$-plane, which takes on analytic values on a portion of this axis. By the preceding theorem, $U$ can be analytically extended across the axis of reals, and on mapping this extended function back on the $(x, y)$-plane we obtain an analytic continuation of $u$ across the curve $AB$. Since $O$ was an arbitrary point of $AB$, this theorem holds along the entire arc.

We note the following special case: If the entire boundary of $D$ is a single analytic curve without singular points (as, for example, a circle), then a regular potential $u$ which takes on

[5]See Chapter VI, Art. 4.

analytic values on the entire boundary can be extended everywhere across the boundary, giving a regular harmonic function in a larger region which contains $D$, together with its boundary, in its interior. However, if the boundary $C$ contains points where $C$ is not analytic, as when $C$ is made of several analytic arcs meeting at angles, these points cannot in general be considered as interior points in an extended region of regularity.

### Art. 9. Green's Function in the Plane

We now come to Green's function. It plays an important role in potential theory, particularly in solving boundary value problems, and in conformal mapping (Art. 12). Concerning its physical meaning (in 3-dimensional space) in electricity, see Chapter IX, Art. 2. The connection between Green's function and the Dirichlet problem is based on equation (44). It can be shown directly that the function defined by the integral in (44) solves the Dirichlet problem (see, for instance, Frank-Mises, "Die Differential- und Integralgleichungen der Mechanik und Physik," Braunschweig, 1930, vol. I, pp. 702-4). But we shall only draw some conclusions from (44), and shall solve the boundary value problems later (Chapter XI) in another way. We now define Green's function.

Let $R$ be a finite plane region with boundary curve $C$. *The Green's function $G$ for the region $R$ is a function of two points $P$:$(x, y)$ and $Q$:$(\xi, \eta)$, of which $P$ varies in $R$ and $Q$ in $R + C$, with the following properties: regarded as a function of $(\xi, \eta)$ with $P$ fixed, it is a regular harmonic function in $R$ (except at $P$) and continuous in $R + C$. It becomes logarithmically infinite as $Q \to P$, in such a way that*

$$(42) \qquad G(x, y; \xi, \eta) - \log \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} = w(x\, y; \xi, \eta)$$

*is a regular harmonic function at $P$ also. It vanishes on the boundary $C$; that is, it has the value zero when $P$ is any point in*

*R and Q is any point of C.* It is, of course, first necessary to prove that the Green's function exists for a region. We assume that $C$ possesses a continuously turning tangent except for a finite number of singular points. The determination of the Green's function is then a special case of the first boundary value problem, since $w$ is a regular harmonic function of $(\xi, \eta)$ in $R$, continuous in $R + C$, which takes on the value $-\log \frac{1}{r}$ on $C$. Since $P$ lies inside $R$, this boundary value is continuous. The existence of Green's function for certain regions now follows from the argument at the end of Chapter VII. (See also Art. 12 on p. 242.)

If $G$ possesses a continuous normal derivative $\frac{\partial G}{\partial n}$ on $C$, the boundary value problems with given continuous boundary values may be solved if $G$ is known; and the general boundary value problem may be reduced to a particular one, that of finding Green's function. To prove this, we may begin with the formulas (for $P$ inside $R$)

$$\oint_C \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds = 0,$$

$$u(x, y) = \frac{1}{2\pi}\oint_C \left\{ \log\frac{1}{r}\frac{\partial u}{\partial n} - u\frac{\partial\left(\log\frac{1}{r}\right)}{\partial n}\right\} ds,$$

which hold for $u$ and $v$ any two potentials regular in $R$ and continuous in $R + C$, possessing continuous normal derivatives. By addition of these, we find

$$(43) \qquad u(x, y) = \frac{1}{2\pi}\oint_C \left\{ \left(\log\frac{1}{r} + v\right)\frac{\partial u}{\partial n} - u\frac{\partial\left(\log\frac{1}{r} + v\right)}{\partial n}\right\} ds.$$

Now let $v$ be the above function $w = G - \log \frac{1}{r}$, which is permissible since the existence and continuity of $\frac{\partial w}{\partial n}$ follows from that of $\frac{\partial G}{\partial n}$; this gives the important formula

$$(44) \qquad u(x, y) = -\frac{1}{2\pi}\oint_C u \frac{\partial G}{\partial n} ds.$$

It is possible to show that the hypothesis of a continuous normal derivative $\frac{\partial u}{\partial n}$ is not required; to do this, it is necessary to investigate more closely some of the properties of the Green's function. Since $w$ is continuous in $R + C$ (and hence in the neighbourhood of $P$) and is accordingly bounded, but $\log \frac{1}{r}$ becomes positively infinite on approach to $P$, the Green's function likewise becomes positively infinite at $P$. Hence, if we surround $P$ by a small circle lying in $R$, $G$ takes on large positive values everywhere on this circle. Since $G$ is zero on the boundary $C$, and $G$ must have its minimum on the boundary, it follows easily that $G$ is positive everywhere in $R$. We will now assume that $R$ is simply-connected. *The equation*

$$G = \text{const.} = c > 0$$

*then represents a closed analytic curve without singular points lying inside $C$ and containing $P$ in its interior.* For, this curve is an equipotential curve, and hence is analytic. It must be a closed curve, since equipotential curves can have no end-points, and because it cannot extend to infinity since it lies in a finite region.[6] Finally, if it did not contain $P$ or possessed

[6]It cannot intersect the boundary $C$, on which $G = O$.

multiple points, there would be a portion of $R$, not containing the point $P$, bounded by an equipotential with $G$ harmonic inside and constant on its boundary; from this it would follow that $G$ is constant in this entire portion (remember the uniqueness theorem) and therefore also throughout $R$, which is not the case.

The curves $G = c$ lie near $C$ when $c$ is small, and near $P$ when $c$ is large. It is evident that the family of curves for $0 < c < \infty$ simply covers the region $R$. In the limiting cases $c \to 0$ and $c \to \infty$, the curves approach the boundary $C$ or shrink toward $P$ respectively. Any such curve $G = c$ separates $R$ into two regions; an inner region where $G > c$ and an outer region where $G < c$. On any curve of the family, $\dfrac{\partial G}{\partial n}$ is therefore everywhere negative.

Apply (43), putting $v = w$, to the region inside one of these curves $G = c$, which may be designated by $C_c$; this gives, since $u$ possesses a continuous normal derivative on this curve,

$$u(x, y) = \frac{1}{2\pi} \oint_{C_c} \left( G \frac{\partial u}{\partial n} - u \frac{\partial G}{\partial n} \right) ds = \frac{c}{2\pi} \oint_{C_c} \frac{\partial u}{\partial n} ds - \frac{1}{2\pi} \oint_{C_c} u \frac{\partial G}{\partial n} ds.$$

But the first integral has the value zero, since $u$ is harmonic in the interior, so that

$$u(x, y) = -\frac{1}{2\pi} \oint_{C_c} u \frac{\partial G}{\partial n} ds.$$

Now let $c \to 0$, and because $u$ and $\dfrac{\partial G}{\partial n}$ are continuous on approaching the boundary $C$ of $R$, we again get (44) without the use of the hypothesis of continuity of $\dfrac{\partial u}{\partial n}$ on the boundary.

Since any hypothesis on $\dfrac{\partial u}{\partial n}$ has been eliminated, equation (44) gives an explicit solution of the first boundary value problem in terms of the Green's function, *if* a solution exists.

We need merely substitute for $u$ in (44) the given boundary value $f$ (continuous). But it still remains to be proved that $-\frac{1}{2\pi}\oint\limits_C f\frac{\partial G}{\partial n}\,ds$ is really a solution of the boundary value problem. We will not investigate this question here.

If the region $R$ is not bounded, and for simplicity is assumed to be the exterior of a simple closed curve $C$, the above definition of the Green's function can be used still; it must be regular at infinity.

As for the hypothesis that $\frac{\partial G}{\partial n}$ is continuous on $C$, this is certainly true if $C$ is a single analytic curve without singular points, as a circle, for example. For, the potential $w$ takes on $C$ the analytic values $-\log\frac{1}{r}$, so that $w$ can be analytically continued (see end of Art. 8) everywhere across $C$. Hence the points of $C$ are inner points of a larger region in which $w$ is a regular harmonic function. In this region, $w$ has continuous derivatives, and hence $\frac{\partial w}{\partial n}$ and $\frac{\partial G}{\partial n}$ are continuous on approaching $C$.[7]

The Green's function is symmetric with respect to $P$ and $Q$; i.e.,

$$G(x, y; \xi, \eta) = G(\xi, \eta; x, y). \tag{45}$$

To prove this, let $P_1:(x_1, y_1)$ and $P_2:(x_2, y_2)$ be any two points in $R$, and exclude the regions around these points by two small circles about them, $K_1$ and $K_2$ of radii $\delta_1$ and $\delta_2$; apply Green's formula to the remainder of the region, using

$$(\xi, \eta) = G(x_1, y_1; \xi, \eta) = G_1; v(\xi, \eta) = G(x_2, y_2; \xi, \eta) = G_2.$$

shall solve the boundary value problems in Chapter XI under certain conditions as to $C$ and $f$. If these conditions are satisfied and $G$ has a continuous normal derivative on $C$, the equation (44), with $u$ replaced by $f$, certainly gives the solution. Moreover, it is obvious that (44) gives the solution if $C$ is a circle (see Art. 1 and Art. 10).

Since $G_1$ and $G_2$ are harmonic in the region, we have

$$\oint_C \left(G_1 \frac{\partial G_2}{\partial n} - G_2 \frac{\partial G_1}{\partial n}\right) ds + \oint_{K_1} \left(G_1 \frac{\partial G_2}{\partial n} - G_2 \frac{\partial G_1}{\partial n}\right) ds + \oint_{K_2} \left(G_1 \frac{\partial G_2}{\partial n} - G_2 \frac{\partial G_1}{\partial n}\right) ds = 0.$$

Here the first integral vanishes, since $G_1$ and $G_2$ vanish on $C$. Now let

$$G_1 = \log \frac{1}{r_1} + w_1, \quad G_2 = \log \frac{1}{r_2} + w_2,$$

where $r_1$ and $r_2$ naturally represent the distances $P_1Q$ and $P_2Q$, respectively. Then

$$\oint_{K_1} \left(G_1 \frac{\partial G_2}{\partial n} - G_2 \frac{\partial G_1}{\partial n}\right) ds = \oint_{K_1} \left(w_1 \frac{\partial G_2}{\partial n} - G_2 \frac{\partial w_1}{\partial n}\right) ds + \oint_{K_1} \left(\log \frac{1}{r_1} \frac{\partial G_2}{\partial n} - G_2 \frac{\partial \left(\log \frac{1}{r_1}\right)}{\partial n}\right) ds.$$

The integrand of the first integral on the right is bounded in the neighbourhood of $P_1$, so the value of this integral approaches 0 as $\delta_1 \to 0$; but from (38) of Chapter III, the second integral has the value $-2\pi G_2(P_1)$. Hence

$$\lim_{\delta_1 \to 0} \oint_{K_1} \left(G_1 \frac{\partial G_2}{\partial n} - G_2 \frac{\partial G_1}{\partial n}\right) ds = -2\pi G_2 \,|^{(\xi,\, \eta) = (x_1,\, y_1)} = -2\pi G(x_2, y_2; x_1, y_1),$$

and similarly,

$$\lim_{\delta_2 \to 0} \oint_{K_2} \left(G_1 \frac{\partial G_2}{\partial n} - G_2 \frac{\partial G_1}{\partial n}\right) ds = +2\pi G_1 \,|^{(\xi,\, \eta) = (x_2,\, y_2)} = 2\pi G(x_1, y_1; x_2, y_2).$$

So that we finally have the result that

$$G(x_1, y_1; x_2, y_2) - G(x_2, y_2; x_1, y_1) = 0,$$

which completes the proof of the symmetry, since $P_1$ and $P_2$ are any points in $R$.

In order to avoid, in the above proof, the need of the hypothesis of a continuous $\dfrac{\partial G}{\partial n}$ on $C$, we may proceed as follows. We use one of the equipotential curves $G_1 = \epsilon > 0$, designated by $C_\epsilon$, in place of $C$, and apply the Green's formula to the region between this curve and the two small circles excluding $P_1$ and $P_2$, and then let $\epsilon \to 0$. We have only to prove that

$$\oint_{C_\epsilon}\left(G_1\frac{\partial G_2}{\partial n} - G_2\frac{\partial G_1}{\partial n}\right) ds \to 0.$$

Now

$$\begin{aligned}\oint_{C_\epsilon} G_1\frac{\partial G_2}{\partial n}\, ds &= \epsilon\oint_{C_\epsilon}\frac{\partial G_2}{\partial n}\, ds \\ &= \epsilon\oint_{C_\epsilon}\frac{\partial \log\dfrac{1}{r_2}}{\partial n}\, ds + \epsilon\oint_{C_\epsilon}\frac{\partial w_2}{\partial n}\, ds \\ &= \epsilon\,(-2\pi) + 0 \\ &= -2\pi\epsilon,\end{aligned}$$

since $w_2$ is harmonic inside $C_\epsilon$.

Let $\mu$ be the maximum of $|G_2|$ on $C_\epsilon$; then since $\dfrac{\partial G_1}{\partial n} < 0$ on $C_\epsilon$,

$$\left|\oint_{C_\epsilon} G_2\frac{\partial G_1}{\partial n}\, ds\right| < \mu\left(-\oint_{C_\epsilon}\frac{\partial G_1}{\partial n}\, ds\right) = 2\pi\mu.$$

Hence

$$\left|\oint_{C_\epsilon}\left(G_1\frac{\partial G_2}{\partial n} - G_2\frac{\partial G_1}{\partial n}\right) ds\right| < 2\pi(\epsilon + \mu).$$

Now when $\epsilon \to 0$, $C_\epsilon \to C$, and $\mu \to 0$ since $G_2$ vanishes on $C$, so that we have proved the desired limit.

From the symmetry it follows that $G$ is also a regular harmonic function of $(x, y)$.

In the above, the symmetry of $G$ in $R$ has been proved. We may now extend the definition of $G$, therefore, by allowing $P$ (as well as $Q$) to lie on $C$ as well as inside $R$. We define $G(P, Q) = 0$ for all points $P$ on $C$ and $Q$ in $R + C$, excluding only the possibility of $P$ and $Q$ coinciding. Then the function $G(P, Q)$ is defined for all distinct pairs of points $P, Q$ in the closed region $R + C$. The property of symmetry holds also in the closed region.

## Art. 10. Green's Function for the Circle

The Green's function for the circle may be explicitly represented by elementary functions. Let the circle have the origin $O$ as centre and the radius $l$. Let the point $P':(x', y')$ be the "reflection" in the circle of the point $P:(x, y)$; i.e., the point with polar coordinates $R'$, $\phi'$ given by the equations

$$R' = \frac{l^2}{R}, \quad \phi' = \phi.$$

Let $r'$ be the distance from $Q$ to $P'$; then for all points $Q$ on the circumference of the circle the ratio $r/r'$ is constant, that is, it is independent of $Q$ and dependent only on $P$. For, on account of the similarity of the triangles $OPQ$ and $OQP'$,

$$\frac{r}{r'} = \frac{R}{l}$$

for all points $Q$ on the circle $C$. Hence the Green's function is given by the expression

$$(46) \quad G(x, y; \xi, \eta) = -\log\frac{r}{r'} + \log\frac{R}{l} = \log\frac{1}{r} - \log\frac{l}{Rr'}.$$

It is easy to check the fact that this function has the defining properties of the Green's function.

The symmetry of $G$ can be made evident as follows. Since

$$x' = \frac{l^2x}{R^2}, \; y' = \frac{l^2y}{R^2},$$

it follows that

$$\begin{aligned} r'R &= R\sqrt{\left(\frac{l^2x}{R^2} - \xi\right)^2 + \left(\frac{l^2y}{R^2} - \eta\right)^2} \\ &= \sqrt{\left(\frac{l^2x}{R} - \xi R\right)^2 + \left(\frac{l^2y}{R} - R\eta\right)^2} \\ &= \sqrt{l^4 - 2l^2(x\xi + y\eta) + R^2(\xi^2 + \eta^2)}. \end{aligned}$$

Hence

$$(46^*) \qquad G(x, y; \xi, \eta) = -\log\sqrt{(x-\xi)^2 + (y-\eta)^2} \\ + \log\sqrt{l^4 - 2l^2(x\xi + y\eta) + (x^2 + y^2)(\xi^2 + \eta^2)} - \log l.$$

The symmetry of the function is evident when written in this form.

The Green's function for the exterior of the circle is

$$G = \log\frac{1}{r} - \log\frac{l}{Rr'}.$$

In this, of course, $P$ lies outside the circle and $P$ inside. That $G$ remains regular when $Q:(\xi, \eta)$ goes to infinity, may be seen as follows: we may write $G$ in the form $G = \log\frac{r'}{r} + \log\frac{R}{l}$. Since $R/l$ is independent of $(\xi, \eta)$, it is sufficient to prove the regularity of the potential $\log\frac{r'}{r} = g$ at infinity. But, as $\xi^2 + \eta^2 \to \infty$ with $P$ fixed, it is evident that $\frac{r'}{r} \to 1$, so that $\lim g = 0$, which is sufficient (see Art. 5). It is immediately evident that $G$ satisfies the other characteristic properties which define the Green's function.

We can now give another derivation of the Poisson integral, by applying (44) to the interior of the circle $C$. The formula is certainly applicable, since the normal derivative of $G$ is obviously continuous on approaching the circle $C$. From

$$G = -\log r + \log r' + \log \frac{R}{l},$$

we find $\dfrac{\partial G}{\partial n} = -\dfrac{\partial \log r}{\partial n} + \dfrac{\partial \log r'}{\partial n}$, since $\log \dfrac{R}{l}$ is independent of $(\xi, \eta)$, or

$$\frac{\partial G}{\partial n} = \frac{\cos(r, n)}{r} - \frac{\cos(r', n)}{r'}.$$

Since

$$R^2 = l^2 + r^2 + 2lr\cos(r, n),$$
$$R'^2 = l^2 + r'^2 + 2lr'\cos(r', n),$$

this gives

$$\frac{\partial G}{\partial n} = \frac{-1}{2l}\left(\frac{l^2 - R^2}{r^2} - \frac{l^2 - R'^2}{r'^2}\right).$$

Now $RR' = l^2$, and $\dfrac{r}{r'} = \dfrac{R}{l}$ since $Q$ is on the circle $C$, from which it follows that

$$\frac{l^2 - R'^2}{r'^2} = \frac{R^2 - l^2}{r^2}.$$

From this we find finally that $\dfrac{\partial G}{\partial n} = \left(\dfrac{-1}{l}\right)\dfrac{l^2 - R^2}{r^2}$, so that

$$u = \frac{-1}{2\pi}\oint_C u\frac{\partial G}{\partial n}\,ds = \frac{1}{2\pi l}\oint_C u\frac{l^2 - R^2}{r^2}\,ds.$$

This is Poisson's integral.

## Art. 11. Green's Function of the Second Kind. Characteristic Function

The Green's function of the second kind $N(P, Q)$, also called the characteristic function, is defined in the following manner

(compare Art. 9): It is a regular harmonic function of $Q:(\xi, \eta)$ except at $P$ in the region $V$, continuous with continuous normal derivative on approaching the boundary $C$. It becomes logarithmically infinite at $P$, in such a way that

$$N(P, Q) - \log\frac{1}{r} = w(P, Q) \tag{47}$$

is a regular harmonic function at $P$ also. On $C$ it satisfies the boundary condition

$$\frac{\partial N}{\partial n} = c. \tag{48}$$

The constant $c$ is not arbitrary, but is determined in the following manner: since $w$ is harmonic everywhere in $V$,

$$\oint_C \frac{\partial w}{\partial n}\, ds = 0,$$

so that

$$\oint_C \frac{\partial N}{\partial n}\, ds = c \oint_C ds = cL = \oint_C \frac{\partial}{\partial n}\left(\log\frac{1}{r}\right) ds,$$

where $L$ is the perimeter of $C$. But

$$\oint_C \frac{\partial}{\partial n}\left(\log\frac{1}{r}\right) ds = \oint_K \frac{\partial}{\partial n}\left(\log\frac{1}{r}\right) ds = -2\pi,$$

where $K$ is a small circle about the centre $P$. Hence $c = -\dfrac{2\pi}{L}$.

The proof of the existence of $N$ is a special case of the second boundary value problem, since $w$, considered as a function of $Q$, is a regular harmonic function in $V$, continuous in $V + C$ with continuous normal derivative, which satisfies the boundary condition

$$\frac{\partial w}{\partial n} = -\frac{2\pi}{L} - \frac{\partial}{\partial n}\left(\log\frac{1}{r}\right).$$

Hence it follows that $N(P, Q)$, in case it exists, is uniquely determined to within an additive constant; this constant, however, may depend on $P$.

Just as in the case of the Green's function in Art. 9, we can prove that the characteristic function $N(P, Q)$ is symmetric, that is,

$$N(P, Q) = N(Q, P). \tag{49}$$

Hence it follows that this function is uniquely determined but for an additive constant independent of both $P$ and $Q$.

If $u(P)$ is a function, harmonic in $V$ and continuous with continuous normal derivative in $V + C$, then

$$u(P) = \frac{1}{2\pi} \oint N(P, Q) \frac{\partial u}{\partial n} ds + \frac{c}{2\pi} \oint uds. \tag{50}$$

If $u$ is sought as the solution of the second boundary value problem with the boundary condition $\dfrac{\partial u}{\partial n} = f$, then this boundary value can be used in the above integral. Since $u$ is only determined up to an additive constant, we can prescribe the condition $\oint_C uds$ 0. Then we have the equation

$$u(P) = \frac{1}{2\pi} \oint_C N(P, Q) f(Q) ds. \tag{51}$$

Thus the second boundary value problem is reduced to the determination of the characteristic function $N(P, Q)$. That means: Equation (51) would give the solution if a solution exists.[8]

## Art. 12. Conformal Mapping and the Green's Function

Let $Z = f(z)$ be an analytic function (single-valued) of the

[8]In Chapter XI we shall prove the existence of the solution under certain conditions as to $C$ and $f$.

complex variable $z$. We assume that it is regular in a neighbourhood of $z = z_0$ and the derivative does not vanish at $z_0$; then the inverse function $z = h(Z)$ is single-valued and analytic regular in the neighbourhood of the corresponding point $Z_0 = f(z_0)$. (This is the analogue of the implicit function theorem for real variables.) Then the region around $z_0$ is mapped in a one-to-one reversible manner on the neighbourhood of $Z_0$ by the function $Z = f(z)$. We will investigate this mapping.

Let $C$ be an arbitrary curve through $z_0$ with continuously turning tangent. By the mapping, this corresponds to a curve $L$ through $Z_0$. Let $z$ and $Z$ be corresponding points on $C$ and $L$, then the derivative $f'(z_0)$ is the limiting value of the difference quotient,

$$f'(z_0) = \lim_{z \to z_0} \frac{Z - Z_0}{z - z_0},$$

and hence

$$Z - Z_0 = (f'(z_0) + \epsilon)(z - z_0),$$

where $\lim\limits_{z \to z_0} \epsilon = 0$. Hence

$$|Z - Z_0| = |f'(z_0) + \epsilon| \; |z - z_0|$$

and

$$\text{arc } (Z - Z_0) = \text{arc } (f'(z_0) + \epsilon) + \text{arc } (z - z_0).$$

(By arc $(z - z_0)$ is meant the angle which the directed line-segment $\overline{z_0 z}$ makes with the positive real axis of the $z$-plane, and arc $(Z - Z_0)$ has the same meaning in the $Z$-plane.) Now when $z \to z_0$ along $C$, $Z \to Z_0$ along $L$; then

$$\lim \text{arc } (z - z_0) = \phi, \; \lim \text{arc } (Z - Z_0) = \psi$$

are the direction angles of the tangents to $C$ and $L$ at $z_0$ and $Z_0$ respectively. On passing to the limit, since $\epsilon \to 0$, we have

$$\psi = \phi + \text{arc} \, f'(z_0).$$

Since $f'(z_0) \neq 0$, arc $f'(z_0)$ has a definite value.

Now $f'(z_0)$ is independent of the curve $C$, being only dependent on $z_0$ (and the function $f$). Hence, if $C_1$ is another curve passing through $z_0$, and $L_1$ is its image or corresponding curve through $Z_0$, then the tangents to these curves have the directions $\phi_1$, $\psi_1$, such that

$$\psi_1 = \phi_1 + \text{arc } f'(z_0).$$

From this it follows that

$$\psi_1 - \psi = \phi_1 - \phi.$$

Hence the two curves $C$, $C_1$ make at $z_0$ the same angle as that made by the corresponding curves $L$, $L_1$ at $Z_0$, both in magnitude and in the sense of rotation. For this reason, the transformation is called "angle-preserving" or "conformal." Hence *an analytic function gives a conformal mapping at every point where its derivative is different from zero.*

We also have

$$\frac{|Z - Z_0|}{|z - z_0|} = |f'(z_0) + \epsilon|,$$

or

$$\lim \frac{|Z - Z_0|}{|z - z_0|} = |f'(z_0)|.$$

The ratio of the lengths $\overline{ZZ_0}$ and $\overline{zz_0}$ has therefore a value independent of $C$ and dependent only on $z_0$, namely $|f'(z_0)|$, in the limit, and the elements of arc-length have of course the same ratio. This can be less exactly expressed as follows: Infinitesimal distances in all directions from $z_0$ are magnified or enlarged by the same factor $|f'(z_0)|$ which may, of course, be smaller than unity. If $z_1$, $z_2$ are two points near $z_0$, and the three points map into $Z_1$, $Z_2$, and $Z_0$ respectively, the triangles $z_1z_2z_0$ and $Z_1Z_2Z_0$ are similar; or more precisely, become more and more nearly similar when $z_1$ and $z_2$ approach $z_0$. Hence

we say that any small region and its map are similar, or that the mapping is a "transformation of similarity" in the small. Evidently all the directions through $z_0$ are in one-to-one correspondence with the directions through $Z_0$.

It is obvious that the inverse map is also conformal, since

$$\frac{dz}{dZ}\bigg|^{Z_0} = \frac{1}{f'(z_0)},$$

and does not vanish. Also, if $Z = f_1(z)$ (with $f'_1(z_0) \neq 0$) gives a conformal map of the $z$-plane onto the $Z$-plane, and $\xi = f_2(Z)$ (with $f'_2(Z_0) \neq 0$) gives a conformal map of the $Z$-plane on the $\xi$-plane, then, by combining these functions, the resulting function

$$\xi = f_2(f_1(z))$$

gives a conformal map of the $z$-plane onto the $\xi$-plane, since

$$\left(\frac{d\xi}{dz}\right)\bigg|^{z_0} = \frac{d\xi}{dZ}\,\frac{dZ}{dz} = f'_2(Z_0)\cdot f_1'(z_0) \neq 0.$$

Examples of conformal mapping are found in all books on theory of functions and in many advanced calculus books. For example, $Z = z + c$, where $c$ is an arbitrary (complex) constant, is a translation. Again, $Z = e^{i\theta}z$, where $\theta$ is a real constant, is a rotation about the origin through the angle $\theta$; and $Z = kz$ (with real $k$) is a similarity transformation, or stretching, from the origin as fixed point. The general linear transformation, which can be written in the form $Z = ke^{i\theta}z + c$, is the result of the combination of a translation, rotation, and enlargement. The function $Z = z^2$ has at the origin the derivative $f'(z) = 2z = 0$, and it is easily seen that the map is not conformal at the origin.

We now pass to our real topic. Let $V$ be a *simply-connected open region* of the $z$-plane (that is, we do not consider the boundary curve $C$ to belong to $V$). Then we are interested

in the problem of mapping this conformally on the interior of the unit circle of the $Z$-plane. This is a problem of mapping *im Grossen* or "in the large", instead of *im Kleinen* or "in the small" (i.e., in the neighbourhood of a point).

To begin with, we will assume that there exists a function analytic in $V$, $Z = f(z)$, which maps the region $V$ on the region $|Z| < 1$ in a one-to-one, reversible, conformal manner. Suppose that the centre of the circle, $Z = 0$, corresponds to the point $z = z_0$, so that $Z_0 = f(z_0) = 0$. The power series expansion of $f(z)$ in the neighbourhood of $z_0$ then has the form

$$f(z) = a_1(z - z_0) + a_2(z - z_0)^2 + \ldots + a_n(z - z_0)^n + \ldots ;$$

for the first term must disappear, since $a_0 = f(z_0) = 0$. On the other hand, $f'(z_0) = a_1 \neq 0$, since if this term were zero the map would not be one-to-one at $z_0$. If we put

$$f(z) = (z - z_0)g(z), \quad g(z) = a_1 + a_2(z - z_0) + \ldots ,$$

then $g(z_0) \neq 0$. Moreover, $g(z)$ cannot vanish anywhere in $V$; for if $g(z_1) = 0$, where $z_1$ is any other point than $z_0$ in $V$, then we would have $f(z_1) = 0$, so that the point $z_1$ would map into $Z = 0$, and the map would not be one-to-one. We now make the claim that

$$G = \log \frac{1}{|f(z)|}$$

is the Green's function of $V$ with the singular point $z_0$.

Evidently $G$ is the real part of the analytic function

$$\log \frac{1}{f(z)} ;$$

for let $f(z) = Re^{i\theta}$, then

$$\log \frac{1}{f(z)} = \log \frac{1}{R} - i\theta, \quad \text{and} \quad \log \frac{1}{|f(z)|} = \log \frac{1}{R} = G.$$

Hence $G$ is certainly harmonic in $V$. Let $H$ be the harmonic function conjugate to $G$, then

$$G + iH = \log \frac{1}{f(z)} = \log \frac{1}{z - z_0} + \log \frac{1}{g(z)} \cdot$$

Moreover,

$$G = \log \frac{1}{|z - z_0|} + \log \frac{1}{|g(z)|}$$

is regular in all $V$ except the point $z_0$ (since $g(z) \neq 0$), and has the characteristic logarithmic singularity of the Green's function at $z = z_0$. Finally, when $z$ approaches the boundary of $V$, then $Z$ approaches the boundary of the unit-circle; therefore $|f(z)| \to 1$ and hence $G \to 0$. Therefore $G(z)$ has all the defining properties of the Green's function. Hence we have proved that: *If the analytic function $Z = f(z)$ maps the region $V$ in a one-to-one reversible manner on the interior of the unit circle, so that $f(z_0) = 0$, then*

$$G = \log \frac{1}{|f(z)|}$$

*is the Green's function of $V$ with the singular point $z_0$.* Evidently $f(z) = e^{-(G+iH)}$.

We will now prove the converse of the above theorem. Let $G(z, z_0)$ be the Green's function for $V$ with the singular point $z_0$, and let $H(z, z_0)$ be the conjugate harmonic function. $G(z, z_0)$ has, by definition, the form

$$G(z, z_0) = \log \frac{1}{r} + u(z, z_0),$$

where $r$ is the distance between the points $z$, $z_0$, and $u$ is harmonic single-valued and regular in $V$. Let $z - z_0 = re^{i\phi}$, and designate the function conjugate to $u$ by $v(z, z_0)$; then

$$G + iH = \log \frac{1}{r} - i\phi + u + iv = \log \frac{1}{z - z_0} - F(z),$$

where we have set $u + iv = -F(z)$, and where $H = -\phi + v$.

Here $v$ as well as $u$ is a single-valued harmonic function. On the other hand, $\phi$ is infinitely multiple-valued and increases monotonely by $2\pi$ when $z$ makes a complete circuit around $z_0$ in the positive direction. Hence $H$ is also infinitely multiple-valued, only determined to within an added multiple of $2\pi$, and decreases by $2\pi$ on a circuit around $z_0$. It decreases monotonely along an equipotential curve $G =$ const. For, on such a curve $\frac{\partial G}{\partial n} < 0$ everywhere (see Art. 9). On the other hand, $\frac{\partial G}{\partial n} = \frac{\partial H}{\partial s}$ (the last symbol meaning the derivative in the positive direction of the curve), since $G$ and $H$ are conjugate functions, so that the Cauchy-Riemann equations are valid. Therefore $\frac{\partial H}{\partial s} < 0$ everywhere on the curve $G =$ const.

After these preparations, the proof is simple. We set

$$Z = f(z) = e^{-G-iH} = (z - z_0)e^{F(z)} = (z - z_0)g(z),$$

where $g(z) = e^{F(z)}$, like $F(z)$, is analytic regular in all $V$, and $g(z) \neq 0$ in $V$ (since the exponential function can vanish for no finite value of its exponent).

Since $G$ is positive throughout $V$ (see Art. 9), it follows that

$$|f(z)| = e^{-G} < 1$$

throughout $V$. Hence to every point $z$ of $V$ there corresponds a point $Z$ interior to the unit circle. Now we must show, conversely, that to every point inside the unit circle corresponds a point of $V$. Let $Z_1 = e^{-a-ib}$ be such a point; since $|Z_1| < 1$, $a > 0$, while $b$ is an arbitrary real number. The point $Z_1$ lies on the circle $|Z_1| = e^{-a}$. This circle is the map of the curve $G = a =$ const. of the $z$-plane, that is, by Art. 9 (because $V$ is supposed to be a simply-connected region), an analytic closed curve containing $z_0$ in its interior. On this curve, since the function $H$ monotonely decreases by $2\pi$ for each revolution,

there is just one point for which $H = b$ (or differs from $b$ by an integral multiple of $2\pi$). So that in fact there corresponds to $Z_1$ in $|Z_1| < 1$ one and only one point $z_1$ of $V$. The map is therefore one-to-one and reversible. Moreover, the point $z = z_0$ corresponds to $Z = 0$, since $G(z_0) = +\infty$. Hence we have the theorem: *If $G$ is the Green's function for $V$ with the singular point $z_0$ and $H$ is its conjugate function, then the analytic function*

$$f(z) = e^{-G-iH}$$

*maps the region $V$ in a one-to-one reversible manner on the interior of the unit circle so that $z_0$ corresponds to its centre.*

The above theorems show the close connection between the conformal mapping of a simply-connected region and the Green's function of the region. The existence proof for the Green's function for such a region and the existence proof for the conformal mapping of $V$ on the interior of the unit circle are therefore completely equivalent. In function-theory, the existence of a function which gives such a map is proved by the methods of the theory of functions, and hence it follows that the Green's function for $V$ exists. We will prove, conversely, the existence of the Green's function for a closed region with boundary having continuous curvature, by solving the first boundary value problem by the method of integral equations; and hence by using the above theorems, give an independent proof of the *fundamental theorem of conformal mapping*, which was first stated by Riemann: *Every simply-connected region bounded by a closed curve with continuous curvature can be conformally mapped in a one-to-one reversible manner on the interior of the unit circle.*

Riemann's theorem was not confined to regions bounded by curves with continuous curvature. We want, therefore, to stress that it is possible to carry out the existence proof for the solution of the first boundary value problem for more general

regions, in particular for every region whose boundary contains more than one point. Accordingly, the fundamental theorem is valid for every simply-connected region whose boundary contains more than one point.

Since there is only one Green's function for $V$ with the singular point $z_0$, the mapping function $f(z) = e^{-G-iH}$ is essentially uniquely determined.[9] However, the function $H$ conjugate to $G$ is only determined to within an added constant. In order to bring out the effect of this, replace $H$ by $H - \gamma$ where $\gamma$ is an arbitrary real constant and $H$ is completely determined. Then

$$Z = f(z) = e^{\gamma i} e^{-G-iH}$$

is the most general mapping function of the desired sort.

This mapping is a combination of the mapping functions $Z_1 = e^{-G-iH}$ and $Z_2 = Z_1 e^{\gamma i}$. The latter is a mere rotation of the $Z$-plane about the origin through the angle $\gamma$, which is arbitrary; it carries the unit circle into itself.

Hence it follows that: there is a unique one-to-one reversible conformal mapping of the simply-connected region $V$ on the interior of the unit circle, which carries a prescribed point $z_0$ into the centre of the circle and a prescribed direction of the $z$-plane at $z_0$ (say the direction of the $x$-axis) into a prescribed direction at the centre of the circle.

Let $V_1$ and $V_2$ be two simply-connected regions. Let the converse of the map described above map the interior of the circle on $V_2$. $V_1$ may be mapped on the circle, and then the circle on $V_2$; and thus, by combining the analytic functions of these maps, we obtain a direct map of $V_1$ on $V_2$. Hence two simply-connected regions may be mapped in a one-to-one reversible conformal manner on each other. The map is unique if a point, and direction through it, of one region

[9]The addition of an integral multiple of $2\pi$ to $H$ does not change $f\ (z)$ because the exponential function has the period $2\pi i$.

is required to go into a prescribed point, and direction through it, in the other region.

The Green's function approaches zero on approaching the boundary. It can be proved that the conjugate function $H$ is likewise continuous on approaching the boundary if the boundary has continuous curvature, or even only a continuously turning tangent. (Of course $H$ is not unique or single-valued on the boundary.) From the continuity of $G$ and $H$, it follows that $f(z)$ is continuous on approaching the boundary, and hence that the boundary of $V$ is likewise mapped in a one-to-one reversible manner on the circumference of the unit circle.

# CHAPTER IX

## THE POISSON INTEGRAL IN SPACE

### Art. 1. Solution of the First Boundary Value Problem for the Sphere. The Harnack Theorems for Space

The solution of the first boundary value problem for the interior of the sphere $S$ with radius $l$ about the origin as centre is given by the Poisson Integral

$$(1) \qquad u = -\frac{1}{2\pi}\iint_S f\left(\frac{\cos(r, n)}{r^2} + \frac{1}{2lr}\right) dS.$$

For, in the first place, $u$ is a regular potential function in the interior, being the sum of the potential of a double layer,

$$V = \iint_S \left(-\frac{f}{2\pi}\right)\frac{\cos(r, n)}{r^2}\, dS,$$

and the potential of a surface distribution,

$$W = \iint_S \left(-\frac{f}{4\pi l}\right)\frac{dS}{r}.$$

And secondly, $u$ approaches the required boundary value when $P:(x, y, z)$ approaches any point $A$ of $S$. For, from (17), Chapter V,

$$V_- = V_A - 2\pi\left(-\frac{f_A}{2\pi}\right) = f_A + \iint_S \left(-\frac{f(Q)}{2\pi}\right)\frac{\cos(r_{QA}, n_Q)}{r^2_{QA}}\, dS,$$

and therefore, since $\cos(r, n) = -\dfrac{r}{2l}$ when $P$ is at a surface point $A$,

$$V_- = f_A + \frac{1}{4\pi l}\iint_S \frac{f(Q)}{r_{QA}}\, dS_Q.$$

Moreover, since $W$ is continuous on approaching $S$,

$$W_{-} = W_A = -\frac{1}{4\pi l}\iint\limits_S \frac{f dS}{r}.$$

Hence

$$u_{-} = V_{-} + W_{-} = f_A.$$

That is, the function $u$ defined by (1) approaches the required boundary value when $P$ approaches the surface from the interior, or along the negative end of the exterior-pointing normal.

The integral (1) can be simplified as follows: we have

$$R^2 = l^2 + r^2 + 2lr \cos(r, n),$$

$$-\left(\frac{\cos(r, n)}{r^2} + \frac{1}{2lr}\right) = \frac{l^2 - R^2}{2lr^3},$$

so that Poisson's integral takes the form

$$u(x, y, z) = \frac{1}{4\pi}\iint\limits_S f\,\frac{l^2 - R^2}{lr^3}\,dS. \tag{2}$$

If we designate the angle $QOP$ by $\alpha$, then

$$r^2 = l^2 + R^2 - 2lR\cos\alpha$$

so that

$$u(x, y, z) = \frac{1}{4\pi l}\iint\limits_S f\,\frac{l^2 - R^2}{(l^2 + R^2 - 2lR\cos\alpha)^{3/2}}\,dS. \tag{2*}$$

Let the polar coordinates of $P$ be $R, \theta, \phi$, where $\theta$ is the angle from the $z$-axis and $\phi$ corresponds to the geographic longitude, so that

$$x = R\sin\theta\cos\phi, \quad y = R\sin\theta\sin\phi, \quad z = R\cos\theta.$$

Let the polar coordinates of $Q$ be $l, \theta', \phi'$. Let $\bar{P}$ with coordinates $l, \theta, \phi$ be the projection of $P$ on the surface $S$ of the sphere, and let $N$ be the intersection of the $z$-axis with the sphere

("north pole"); then the law of cosines of spherical trigonometry applied to the spherical triangle $\bar{P}QN$ gives

(3) $$\cos\alpha = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi').$$

This expression for $\cos\alpha$ is to be substituted in (2*).

We can write (2*) in the form

(4) $$u = \frac{l}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{f(\theta',\phi')\,(l^2 - R^2)\sin\theta'}{(l^2 + R^2 - 2lR\cos\alpha)^{3/2}}\,d\theta' d\phi'.$$

The formula (4), combined with (3), is the starting-point for obtaining the expansion of $u$ in spherical harmonics, which is the analogue of the expansion of the logarithmic potential in trigonometric functions obtained in the preceding chapter.

The Poisson integral for the exterior of the sphere is

(5) $$u = \frac{1}{2\pi}\iint_S f\left(\frac{1}{2lr} - \frac{\cos(r,n)}{r^2}\right)dS.$$

It solves the second formulation of the exterior boundary value problem (see Chapter VII, Art. 3); for in the first place $u$ is regular outside the sphere, also at infinity, being the sum of potentials of a surface distribution and of a double layer on the sphere, and secondly, $u_+ = f_A$, as may be easily proved from the results in Chapter V, Arts. 4 and 5. The mass of the above solution is evidently

(6) $$M = \frac{1}{4\pi l}\iint_S f dS.$$

*The Harnack theorems,* now that the Poisson integral has been derived, *may be taken over word for word for space from the corresponding theorems for logarithmic potential.*

## Art. 2. Green's Function in Space

Let $V$ be a finite region of space bounded by the closed surface $S$, having a continuously turning tangent plane except for a

finite number of edges and conical points. *The Green's function for $V$ is a function $G$ of two points $P$:$(x, y, z)$ and $Q$:$(\xi, \eta, \zeta)$, of which $P$ lies in $V$ and $Q$ in $V + S$, with the following properties: As a function of $(\xi, \eta, \zeta)$ it is a regular potential in $V$ except at $P$, continuous in $V + S$ and vanishing on $S$. It becomes infinite at $P$ like the reciprocal of the distance $QP$, in such a way that*

$$(7) \qquad w(x, y, z; \xi, \eta, \zeta) = G(x, y, z; \xi, \eta, \zeta) - \frac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}$$

*is a regular potential function at $P$ also.*

The determination of the Green's function is a special case of the first boundary value problem, since $w$ must take on the value $-1/r$ on the boundary.

If the Green's function has a continuous normal derivative on $S$ and if $u$ is a regular potential in $V$, continuous in $V + S$, then

$$u = -\frac{1}{4\pi}\iint_S u \frac{\partial G}{\partial n}\, dS.$$

The Green's function in space likewise has the property of symmetry,

$$G(x, y, z; \xi, \eta, \zeta) = G(\xi, \eta, \zeta; x, y, z).$$

The Green's function for the interior of a sphere $S$ about the origin of radius $l$ is

$$(8) \qquad G = \frac{1}{r} - \frac{l}{R}\cdot\frac{1}{r'},$$

where $r'$ is the distance from $Q$ to the point $P'$ (the inverse point or reflection of $P$ in $S$) which lies on the line $OP$ extended, at the radius $R' = l^2/R$ from $O$. The fact that $G$ vanishes on

the surface of the sphere follows from the relation

$$\frac{r}{r'} = \frac{R}{l},$$

just as in the case in the plane. We will again bring $G$ into a form where its symmetry is evident. We have

$$x' = l^2x/R^2,\ y' = l^2y/R^2,\ z' = l^2z/R^2,$$

$$Rr' = \sqrt{\left(\frac{l^2x}{R} - R\xi\right)^2 + \left(\frac{l^2y}{R} - R\eta\right)^2 + \left(\frac{l^2z}{R} - R\zeta\right)^2},$$

so that

$$(8^*)\quad G = \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}} - \frac{l}{\sqrt{l^4 - 2l^2(x\xi + y\eta + z\zeta) + (x^2+y^2+z^2)(\xi^2+\eta^2+\zeta^2)}}$$

The Green's function for the exterior of the sphere is

$$G = \frac{1}{r} - \frac{l}{Rr'}.$$

The reader should show that this behaves at infinity like a potential of mass $\left(1 - \frac{l}{R}\right)$.

Give the general definition of the Green's function for the exterior of a closed surface.

The physical meaning of the Green's function is easily shown. If $S$ is a grounded electrical conductor and if the unit charge is concentrated at $P$, then

$$G(P, Q) = \frac{1}{r} + w(P, Q)$$

is the value at $Q$ of the potential due to the charge at $P$ and the induced charge on $S$ (remember that $G$ vanishes on $S$). The function $\frac{1}{r}$ is the potential of the unit charge, and $w$ is the potential of the induced charge.

## Art. 3. Expansion of a Harmonic Function in Spherical Harmonics

We will now obtain the expansion of an arbitrary potential or harmonic function in a series of spherical harmonics (compare Chapter IV, Art. 3). Let $u(\rho, \theta, \phi)$ be harmonic and regular in the interior of the sphere $S$ and continuous on approaching $S$, and let $u(l, \theta, \phi) = f(\theta, \phi)$. Then $u$ can be represented by the Poisson integral (2),

$$u(\rho, \theta, \phi) = \frac{1}{4\pi l}\iint_S f(\theta', \phi') \frac{l^2 - \rho^2}{r^3}\, dS.$$

First we will expand the integrand,

$$\frac{l^2 - \rho^2}{r^3} = -\frac{1}{r} - 2l \frac{\partial\left(\frac{1}{r}\right)}{\partial n}.$$

Since $\frac{\partial}{\partial n}$ denotes the derivative in the direction of the outward normal or of the radius, we can consider this derivative as differentiation with respect to $l$. We then obtain from Chapter IV, Art. 2, for $\rho < l$, if

$$\cos \alpha = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi'),$$

$$\frac{1}{r} = P_0(\cos \alpha)\frac{1}{l} + P_1(\cos \alpha)\frac{\rho}{l^2} + \ldots + P_m(\cos \alpha)\frac{\rho^m}{l^{m+1}} + \ldots,$$

and by term-wise differentiation with respect to $l$, with $\mu = \cos \alpha$,

$$\frac{\partial\left(\frac{1}{r}\right)}{\partial n} = \frac{\partial\left(\frac{1}{r}\right)}{\partial l} = -P_0(\mu)\frac{1}{l^2} - 2P_1(\mu)\frac{\rho}{l^3} - \ldots - (m+1)P_m(\mu)\frac{\rho^m}{l^{m+2}} - \ldots$$

Hence

$$\frac{l^2 - \rho^2}{r^3} = \sum_{m=0}^{\infty} (2m + 1)P_m(\mu)\frac{\rho^m}{l^{m+1}}, \tag{9}$$

so that

$$u(\rho, \theta, \phi) = \sum_{m=0}^{\infty} \frac{2m+1}{4\pi l^2} \cdot \frac{\rho^m}{l^m} \iint\limits_S f(\theta', \phi')P_m(\mu)\, dS. \tag{10}$$

The integral $\iint\limits_S f(\theta', \phi')P_m(\mu)dS$ is a surface spherical harmonic,

and $$\rho^m \iint\limits_S fP_m dS$$

is a spherical harmonic of order $m$. Eqn. (10) gives therefore the desired expansion. It is valid for $\rho < l$, or in the interior of the sphere, and uniformly convergent there.

For $\rho = l$ the series on the right becomes

$$\sum^{\infty} \frac{2m+1}{4\pi} \iint f(\theta', \phi')P_m(\mu)d\omega$$

$$(d\omega = \sin\theta' d\theta' d\phi';\ dS = l^2 d\omega)$$

where the integration is to be carried out over the unit sphere. It is called the "Laplace Series." In the following paragraphs we will study the conditions under which the expansion (10) is also valid on the surface of the sphere.

We will now obtain an important consequence of (10). Let $u$ itself be an arbitrary spherical harmonic of order $n$, and therefore $u = \rho^n A_n(\theta, \phi)$ where $A_n$ is an arbitrary surface spherical harmonic. Then

$$\rho^n A_n(\theta, \phi) = \sum_{m=0}^{\infty} \frac{2m+1}{4\pi l^2} \cdot \frac{\rho^m}{l^m} \iint\limits_S l^n A_n(\theta', \phi')P_m(\mu)dS.$$

The coefficients of like powers of $\rho$ must agree on the two sides of this equation, so that

$$(11) \qquad \iint A_n(\theta', \phi')P_m(\mu)d\omega = 0 \text{ for } n \neq m,$$

and

$$(12) \qquad A_n(\theta, \phi) = \frac{2n+1}{4\pi}\iint A_n(\theta', \phi')P_n(\mu)d\omega.$$

The orthogonality property (11) is a special case of (37) of Chapter IV, Art. 6.

The equation (12) leads to further results. From the addition theorem, $P_n(\cos \alpha)$ can be expressed as a linear combination of the functions $P_n(\cos\theta)$, $P_n^\nu(\cos\theta)\cos\nu\phi$ and $P_n^\nu(\cos\theta)\sin\nu\phi$. Therefore from (12) we obtain an expansion for $A_n(\theta, \phi)$ itself,

$$(13) \qquad A_n(\theta, \phi) = a_0 P_n(\cos\theta) + \sum_{\nu=1}^{n} P_n^\nu(\cos\theta)\left\{a_\nu \cos\nu\phi + b_\nu \sin\nu\phi\right\},$$

where $a_0, a_\nu, b_\nu$ are constants. Accordingly, any surface spherical harmonic of order $n$ can be represented as a linear homogeneous combination with constant coefficients of the above $2n+1$ functions. The above functions are evidently linearly independent. *Therefore there are exactly $2n+1$ linearly independent spherical harmonics of order $n$.*

If $f(\theta', \phi')$ is a continuous single-valued function over the unit sphere $E$, then $\iint_E f(\theta', \phi')P_n(\mu)\sin\theta' d\theta' d\phi'$ is certainly a surface spherical harmonic of order $n$. It follows from (12) that *the most general surface spherical harmonic* can be represented in this way.

If the harmonic function $u$ is regular in the neighbourhood of a point, which we may choose as the origin $O$, then we may choose $l$ so small that the sphere $S$ of radius $l$ lies in the region of regularity of $u$. Then $u$ can be expanded in the form

$$(10^*) \qquad u(\rho, \theta, \phi) = \sum_{m=0}^{\infty} \frac{2m+1}{4\pi l^2} \frac{\rho^m}{l^m} \iint u(l, \theta', \phi')P$$

and this expansion is certainly valid for $\rho < l$.

If the expansion of a harmonic function $u$ is already given in a uniformly convergent series of spherical harmonics of the form

$$u(\rho, \theta, \phi) = \sum_{m=0}^{\infty} \rho^m A_m(\theta, \phi),$$

valid in the interior of the sphere $S$ of radius $l$, let $c$ be any positive number $<l$. Then, since the expansion is valid for $\rho = c$, we have for the determination of the $A_n(\theta, \phi)$ (with the use of (11) and (12))

$$\iint_E u(c, \theta', \phi')P_m(\mu)d\omega = c^m \iint_E A_m(\theta', \phi')P_m(\mu)d\omega$$
$$= \frac{4\pi c^m}{2m+1} A_m(\theta, \phi),$$

so that

$$A_m(\theta, \phi) = \frac{2m+1}{4\pi c^m} \iint_E u(c, \theta', \phi')P_m(\mu)d\omega.$$

We find for $u$, therefore, the expansion

$$u = \sum_{m=0}^{\infty} \frac{2m+1}{4\pi} \frac{\rho^m}{c^m} \iint_E u(c, \theta', \phi')P_m(\mu)d\omega$$

where $c$ is an arbitrary positive constant $<l$. If the function $u$ is also defined on $S$ and continuous on approaching $S$, or in $V + S$, with the limiting value

$$\lim_{c \to l} u(c, \theta', \phi') = f(\theta', \phi'),$$

then on passing to the limit, which is easily shown to be possible in every term, we obtain again the expansion (10).

If the series (10) is convergent for a point $P$ of $S$, then the equation (10) holds for this point, by Abel's theorem. If the

series in (10) converges for every point of $S$, i.e. if the Laplace series converges everywhere on $S$, then it represents everywhere on $S$ the function $u(l, \theta, \phi) = f(\theta, \phi)$, and $f$ is developable in surface spherical harmonics. In the following paragraphs we will investigate the development of a function in a Laplace series independently of the Poisson integral.

*Exercise*: Show that if $u$ is regular in the space between two concentric spheres, one gets an expansion similar to that of Chapter VIII, Art. 5.

## Art. 4. Expansion of an Arbitrary Function in Surface Spherical Harmonics

*A function $f(\theta, \phi)$ which is continuous and single-valued on the unit sphere $E$, with continuous first derivatives, may be expanded in a Laplace series, and this series is uniformly convergent over $E$.* That is,

$$f(\theta, \phi) = \sum_{m=0}^{\infty} \frac{2m+1}{4\pi} \iint_E f(\theta', \phi') P_m(\mu) d\omega, \tag{14}$$

$$\mu = \cos \alpha = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi - \phi').$$

Of course, $(\theta, \phi)$ is an arbitrary point on $E$.

The assumption of continuous first derivatives means that the function has continuous partial derivatives $\frac{\partial f}{\partial \theta}$ and $\frac{\partial f}{\partial \phi}$ with respect to every system of coordinates $\theta$, $\phi$ of $E$. We may therefore choose the system of $\theta$, $\phi$ at liberty according to our purposes.

To prove the above theorem, we first form the partial sum (as in Fourier series)

$$s_n(\theta, \phi) = \sum_{m=0}^{n} \frac{2m+1}{4\pi} \iint_E f(\theta', \phi') P_m(\cos\alpha)\, d\omega$$
$$= \frac{1}{4\pi} \iint_E f(\theta', \phi') \left\{ \sum_{m=0}^{n} (2m+1) P_m(\cos\alpha) \right\} d\omega.$$

By using the relation

$$P_n'(\mu) = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} (2n - 4k - 1)\, P_{n-2k-1}(\mu)$$

obtained in (15) of Chapter IV, Art. 2, the sum in the bracket above may be given a simpler form. Evidently

$$(15) \qquad \sum_{m=0}^{n} (2m + 1) P_m(\mu) = P_n'(\mu) + P_{n+1}'(\mu).$$

This identity plays a similar role to that played by (18) of Chapter VIII in the proof of the convergence of Fourier series. It follows that

$$(16) \qquad s_n(\theta, \phi) = \frac{1}{4\pi} \iint_E f(\theta', \phi') \left\{ P_n'(\cos \alpha) + P_{n+1}'(\cos \alpha) \right\} d\omega.$$

We can now (without limiting the generality of the investigation) so choose the coordinate system that the point $P: (\theta, \phi)$ is the "north pole"; then $\theta = 0$ and $\cos \alpha = \cos \theta'$, so that

$$s_n(P) = \frac{1}{4\pi} \iint_E f(\theta', \phi') \left\{ P_n'(\cos \theta') + P_{n+1}'(\cos \theta') \right\} d\omega,$$

or if we introduce the abbreviation

$$(17) \qquad \int_0^{2\pi} f(\theta', \phi') d\phi' = g(\theta') = h(v) \quad [v = \cos \theta'],$$

then

$$\begin{aligned} s_n(P) &= \frac{1}{4\pi} \int_0^{\pi} g(\theta') \left\{ P_n'(\cos \theta') + P_{n+1}'(\cos \theta') \right\} \sin \theta' d\theta' \\ &= \frac{1}{4\pi} \int_{-1}^{1} h(v) \left\{ P_n'(v) + P_{n+1}'(v) \right\} dv. \end{aligned}$$

Hence, by integration by parts,

$$\begin{aligned} 4\pi s_n(P) = h(v) &\left\{ P_n(v) + P_{n+1}(v) \right\} \Big|_{v=-1}^{v=+1} \\ &- \int_{-1}^{+1} h'(v) \left\{ P_n(v) + P_{n+1}(v) \right\} dv. \end{aligned}$$

The integrated term on the right is evidently $2h(1)$. But from (17),

$$h(1) = \int_0^{2\pi} f(0, \phi')d\phi' = f(P) \int_0^{2\pi} d\phi' = 2\pi f(P),$$

because $f(0, \phi')$ means the value of $f$ at the "north pole" $P$ and is therefore independent of $\phi'$. Finally, from this we find

$$(18) \quad s_n(P) - f(P) = -\frac{1}{4\pi} \int_{-1}^{+1} h'(v) \left\{ P_n(v) + P_{n+1}(v) \right\} dv.$$

From (17)

$$h'(v) = \frac{dh}{dv} = -\frac{dg}{d\theta'} \cdot \frac{1}{\sin\theta'} = -\frac{1}{\sin\theta'} \int_0^{2\pi} \frac{\partial f(\theta', \phi')}{\partial \theta'} d\phi'.$$

Now the derivative $\dfrac{\partial f}{\partial \theta'}$ is continuous on $E$ and therefore is bounded, say

$$(19) \qquad \left|\frac{\partial f}{\partial \theta'}\right| < \frac{M}{2\pi}; \text{ then} \left|\frac{dh}{dv}\right| < \frac{M}{\sin\theta'} = \frac{M}{\sqrt{1 - v^2}}$$

for all points of $E$ (that means: $h'(v)\sqrt{1 - v^2}$ is bounded). From this it can be proved that the integral (18) can be made as small as desired in absolute value by taking $n$ sufficiently large. Let $c$ be a number near 1, and write

$$\int_{-1}^{1} h'(v) P_n(v) dv = \int_{-1}^{-c} \cdots + \int_{-c}^{c} \cdots + \int_{c}^{1} \cdots.$$

Now (remember $|P_n(v)| \leq 1$)

$$\left|\int_c^1 h'(v) P_n(v) dv\right| < M \int_c^1 \frac{dv}{\sqrt{1 - v^2}} = M \arc\cos v \Big]_{v=c}^{v=1},$$

so that this can be made less than any positive $\epsilon$ by taking $c$ sufficiently near 1, and this uniformly in $n$. Then it is also true that

$$\left|\int_{-1}^{-c} \cdots\right| < M \arc\cos v \Big]_{v=-1}^{v=-c} < \epsilon.$$

Again,
$$\left|\int_{-c}^{c} h'(v)P_n(v)dv\right| < M\int_{-c}^{c}\frac{|P_n(v)|}{\sqrt{1-v^2}}dv$$
$$< \frac{M}{\sqrt{1-c^2}}\int_{-1}^{1}|P_n(v)|\,dv$$
$$\leq \frac{M}{\sqrt{1-c^2}}\,2\sqrt{\frac{2}{2n+1}},$$
since
$$\int_{-1}^{1}|P_n(v)|\,dv \leq 2\sqrt{\int_{-1}^{1}P_n^2(v)dv}$$
from the Schwarz inequality (Chapter VIII, Art. 4). After $c$ has been chosen, $n$ can be chosen large enough so that this part of the integral also has a value smaller than $\epsilon$. Then we have $\left|\int_{-1}^{1} h'(v)P_n(v)dv\right| < 3\epsilon$, and likewise of course
$$\left|\int_{-1}^{1} h'(v)P_{n+1}(v)dv\right| < 3\epsilon,$$
so that finally
$$\left|s_n(P) - f(P)\right| < \frac{6\epsilon}{4\pi} < \epsilon.$$
Since the upper bound $M$ in (19) holds for all points of $E$, the choice of $c$ and finally of $n$ holds uniformly over all $E$. Hence the equation
$$\lim_{n\to\infty} s_n(P) = f(P)$$
holds uniformly over $E$, completing the proof.

### Art. 5. Expansion in Legendre Polynomials

The problem of the expansion of an arbitrary function in a series of Legendre polynomials can now be easily treated. First we know from Chapter IV, Art. 6, that if a function $F(u)$ can be expanded in a uniformly convergent series of Legendre polynomials in the interval $-1 \leq u \leq 1$, the expansion must have the form

$$(20) \qquad F(u) = \sum_{m=0}^{\infty} \frac{2m+1}{2} P_m(u) \int_{-1}^{1} F(v) P_m(v) dv.$$

For $u = \cos\theta$, $v = \cos\theta'$ this becomes

$$F(\cos\theta) = \sum_{m=0}^{\infty} \frac{2m+1}{2} \int_0^{\pi} F(\cos\theta') P_m(\cos\theta) P_m(\cos\theta') \sin\theta' d\theta',$$

and hence by the use of (45) of Chapter IV, Art. 7,

$$(21) \qquad F(\cos\theta) = \sum_{m=0}^{\infty} \frac{2m+1}{4\pi} \iint_E F(\cos\theta') P_m(\cos\alpha) d\omega,$$

$$\cos\alpha = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\phi, \quad d\omega = \sin\theta' d\theta' d\phi.$$

But the expansion (21) is a special case of the expansion (14), in which $f(\theta, \phi)$ is replaced by the function $F(\cos\theta)$ independent of $\phi$. If the function $F(u)$ has a continuous derivative in $-1 \leq u \leq 1$, then $F(\cos\theta)$ has a continuous derivative over the unit sphere; hence the expansion (21) or (20) is justified. Hence:

*An arbitrary function which has a continuous first derivative in the interval $-1 \leq u \leq 1$, may be expanded in a Legendre polynomial series; the series converges uniformly in the interval.*

The subject of expansion in a series of Legendre polynomials can also be treated in other ways. We will merely state here without proof the result of W. H. Young:[1] If the Legendre series of the function $F(u) = F(\cos\theta)$ has the partial sum $s_n(\cos\theta)$, and the Fourier series for $\sin\theta\, F(\cos\theta)$ has the partial sum $\sigma_n(\cos\theta)$, then $\lim_{n\to\infty} \left\{ s_n(\cos\theta) - \frac{\sigma_n(\cos\theta)}{\sin\theta} \right\} = 0$ holds for $-1 < u < 1$ or $0 < \theta < \pi$. From this it follows that if the function $\sin\theta\, F(\cos\theta)$ satisfies such conditions that it can be expanded in a Fourier series, then $F(u)$ can be expanded in a Legendre polynomial series in $-1 < u < 1$. This says, however, nothing about the end-points.

[1] Comptes Rendus, vol. 165, "Sur les series de polynomes de Legendre."

# CHAPTER X

## THE FREDHOLM THEORY OF INTEGRAL EQUATIONS

We have solved the boundary value problem only for the circle and the sphere, and even for these very special regions we have solved only the first boundary value problem.

Of the many methods which may be used to solve these problems in general, we will choose the Fredholm method of integral equations, which is distinguished because of its elegance and power. In this chapter we will develop the fundamental results of the Fredholm theory.

### Art. 1. The Problem of Integral Equations

An integral equation is a functional equation, in which the unknown function appears under the integral sign.

The most important types of integral equations are

$$\int_a^b K(x, \xi)\phi(\xi)d\xi = f(x) \tag{A}$$

and

$$\phi(x) - \lambda\int_a^b K(x, \xi)\phi(\xi)d\xi = f(x). \tag{B}$$

Here $K(x, \xi)$ and $f(x)$ are given, $\phi(x)$ is the desired unknown function, and $\lambda$ is a parameter.

The variables $x$ and $\xi$ are limited to the real interval $a$ to $b$. The functions $K(x, \xi)$ and $f(x)$ may take on complex values, or in other words may be complex functions of real arguments. The parameter $\lambda$ may also be complex.

The function $K(x, \xi)$ is known as the *kernel* of the integral equation. The equation (A), in which the unknown function occurs only under the integral sign, is known as an *integral equation of the first kind*. The equation (B), in which the

unknown function $\phi(x)$ occurs both under the integral sign and outside it, is an *integral equation of the second kind.* When $f(x)$ does not vanish identically in $a \leqq x \leqq b$, the equation (B) is called a *non-homogeneous equation,* while for $f(x) \equiv 0$, it becomes

$$\text{(C)} \qquad \phi(x) - \lambda \int_a^b K(x, \xi)\phi(\xi)d\xi = 0,$$

which is called the *corresponding homogeneous equation.*

The above integral equations are all *linear* with respect to the unknown function $\phi(x)$, since it enters everywhere to the first degree. (We will not consider non-linear equations.)

Such equations may be formed for two-, three-, or multi-dimensional regions also. It is permissible that the region of integration be a curved surface (or space) with proper curvilinear coordinates to locate points uniquely in the region.

The equation

$$u(x, y) = \iint_R \rho(\xi, \eta) \log \frac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2}} d\xi d\eta$$

(with $R$ meaning a region of the plane), which defines the logarithmic potential of a distribution, may be regarded as an integral equation if $u$ is given and $\rho$ is the unknown. This is an example of an integral equation of the first kind with the kernel

$$\log \frac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2}} \cdot$$

Its solution is given by Poisson's equation,

$$\rho(x, y) = -\frac{\nabla^2 u}{2\pi} \cdot$$

Similarly, the equation

$$u(x, y, z) = \iiint_R \frac{\rho(\xi, \eta, \zeta)}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}} d\xi d\eta d\zeta \cdot$$

may be regarded as an integral equation of the first kind, with the kernel

$$K(x, y, z; \xi, \eta, \zeta) = \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}};$$

it has the solution

$$\rho(x, y, z) = -\frac{\nabla^2 u}{4\pi}.$$

In the following pages we will be concerned mostly with integral equations of the second kind, for which the theory was founded[2] by Fredholm.[3]

## Art. 2. The First Theorem of Fredholm

We will now consider the functional equation for $\phi(x)$,

$$\text{(I)} \qquad \phi(x) - \lambda \int_a^b K(x, \xi)\phi(\xi)d\xi = f(x).$$

For the present we assume that $f(x)$ and $K(x, \xi)$ are continuous for $a \leqq x \leqq b$ and for $a \leqq x \leqq b$, $a \leqq \xi \leqq b$ respectively. It will be possible to find necessary and sufficient conditions for the existence and uniqueness of a solution $\phi(x)$, and to obtain an explicit solution.

Following Fredholm, we define for $n = 1, 2, 3, \ldots,$

$$\text{(1)} \quad K\begin{pmatrix} x_1, x_2, \ldots, x_n \\ y_1, y_2, \ldots, y_n \end{pmatrix} = \begin{vmatrix} K(x_1, y_1) & K(x_1, y_2) & \ldots & K(x_1, y_n) \\ K(x_2, y_1) & K(x_2, y_2) & \ldots & K(x_2, y_n) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ K(x_n, y_1) & K(x_n, y_2) & \ldots & K(x_n, y_n) \end{vmatrix};$$

[2]For later formulations of the theory see, for instance, Bôcher, *An Introduction to the Study of Integral Equations*, Cambridge, 1909 and 1914; Heywood and Fréchet, *L'équation de Fredholm et ses applications à la physique mathématique*, Paris, 1912; Courant-Hilbert, *Methoden der Mathematischen Physik*, Bd. I, in particular p. 128 ff., Berlin, 1924.

[3]See especially *Acta Math.*, vol. 27.

$$(2)\quad \begin{cases} C_n = \dfrac{1}{n!}\displaystyle\int_a^b \cdots \int_a^b K\begin{pmatrix}\xi_1, \xi_2, \dots, \xi_n\\ \xi_1, \xi_2, \dots, \xi_n\end{pmatrix} d\xi_1\, d\xi_2 \dots d\xi_n, \\ C_0 = 1\,; \end{cases}$$

$$(3)\quad \begin{cases} C_n(x, y) = \dfrac{1}{n!}\displaystyle\int_a^b \cdots \int_a^b K\begin{pmatrix}x, \xi_1, \xi_2, \dots, \xi_n\\ y, \xi_1, \xi_2, \dots, \xi_n\end{pmatrix} d\xi_1\, d\xi_2 \dots d\xi_n, \\ C_0(x, y) = K(x, y). \end{cases}$$

The quantities $C_n$ and $C_n(x, y)$ are always to be considered as having the value zero when the index is negative.

Expanding the determinant $K\begin{pmatrix}x, \xi_1, \xi_2, \dots, \xi_n\\ y, \xi_1, \xi_2, \dots, \xi_n\end{pmatrix}$ by elements of the first line, we have

$$(4)\quad \begin{aligned} K\begin{pmatrix}x, \xi_1, \xi_2, \dots, \xi_n\\ y, \xi_1, \xi_2, \dots, \xi_n\end{pmatrix} &= K(x, y) K\begin{pmatrix}\xi_1, \xi_2, \dots, \xi_n\\ \xi_1, \xi_2, \dots, \xi_n\end{pmatrix} \\ &+ \sum_{k=1}^{n} (-1)^k K(x, \xi_k) K\begin{pmatrix}\xi_1, \xi_2, \dots & & , \xi_n\\ y, \xi_1, \dots, \xi_{k-1}, & \xi_{k+1}, \dots & , \xi_n\end{pmatrix}. \end{aligned}$$

Divide by $n!$ and integrate with respect to $\xi_1, \dots, \xi_n$ from $a$ to $b$, obtaining

$$(5)\quad C_n(x, y) = C_n\, K(x, y) - \int_a^b K(x, \xi) C_{n-1}(\xi, y)\, d\xi$$

for $n = 0, 1, 2, \dots$. For, when $n = 0$ we have simply $K(x, y) = K(x, y)$. For $n > 0$, we note in addition to (2) and (3), that the following identity holds for every $k = 1, 2, \dots n$.

$$\begin{aligned} &K\begin{pmatrix}\xi_1, \xi_2, \dots & & \xi_n\\ y, \xi_1, \dots, \xi_{k-1}, & \xi_{k+1}, \dots, & \xi_n\end{pmatrix} \\ &\qquad = (-1)^{k-1} K\begin{pmatrix}\xi_k, \xi_1, \dots, \xi_{k-1}, \xi_{k+1}, \dots, \xi_n\\ y, \xi_1, \dots, \xi_{k-1}, \xi_{k+1}, \dots, \xi_n\end{pmatrix} \end{aligned}$$

Hence

$$\frac{(-1)^k}{n!}\iint \cdots \int K(x, \xi_k) K\begin{pmatrix}\xi_1, \xi_2, \dots & & , \xi_n\\ y, \xi_1, \dots, \xi_{k-1}, & \xi_{k+1}, \dots & , \xi_n\end{pmatrix} d\xi_1 d\xi_2 \dots d\xi_n$$

$$= \frac{-1}{n!}\int K(x,\xi_k)d\xi_k\int\ldots\int K\begin{pmatrix}\xi_k, \xi_1, \xi_2, \ldots, \xi_{k-1}, \xi_{k+1}, \ldots, \xi_n \\ y, \xi_1, \xi_2, \ldots, \xi_{k-1}, \xi_{k+1}, \ldots, \xi_n\end{pmatrix}$$
$$d\xi_1 \ldots d\xi_{k-1} d\xi_{k+1} \ldots d\xi_n$$
$$= -\frac{1}{n}\int K(x,\xi_k)C_{n-1}(\xi_k, y)d\xi_k$$
$$= -\frac{1}{n}\int K(x,\xi)C_{n-1}(\xi, y)d\xi.$$

Multiply both sides of the identity (5), just proved, by $(-\lambda)^n$ and sum; letting

$$D(\lambda) = \sum_{n=0}^{\infty} C_n(-\lambda)^n \tag{6}$$

and

$$D\binom{x}{y};\lambda\Big) = \sum_{n=0}^{\infty}(-\lambda)^n C_n(x,y), \tag{7}$$

this gives

$$\sum_{n=0}^{\infty}(-\lambda)^n C_n(x,y) = K(x,y)\sum_{n=0}^{\infty}(-\lambda)^n C_n$$
$$+\lambda\int_a^b\left\{\sum_{n=0}^{\infty}(-\lambda)^{n-1}C_{n-1}(\xi,y)\right\}K(x,\xi)\,d\xi$$

or

$$D\left(\begin{matrix}x\\y\end{matrix};\lambda\right) = K(x,y)D(\lambda) + \lambda\int_a^b K(x,\xi)D\left(\begin{matrix}\xi\\y\end{matrix};\lambda\right)d\xi. \tag{8}$$

We will shortly prove that both the series (6) and (7) are *always convergent*, hence that $D(\lambda)$ and $D\left(\begin{matrix}x\\y\end{matrix};\lambda\right)$ are *integral functions* of $\lambda$[4]. The function $D(\lambda)$ is called the "*determinant*" of the kernel $K(x,y)$, and $D\left(\begin{matrix}x\\y\end{matrix};\lambda\right)$ its "*minor determinant of the first order.*"

[4]An analytic function which is regular everywhere except at infinity is called an "integral function." Its expansion in a power series is convergent for every finite value of its argument.

These names are based on the fact that $D(\lambda)$ plays a similar part in the solution of (I) to that played by the determinant of the coefficients in the solution of linear algebraic equations. Similarly, $D\left(\begin{matrix}x\\y\end{matrix}; \lambda\right)$ plays a part similar to that played by the minors of the determinant in the solution of algebraic linear equations by Cramer's rule.

We must now distinguish two cases, depending on whether the determinant $D(\lambda)$ is zero or different from zero for the value of the parameter $\lambda$ for which we wish to solve (I); in this article we will consider the case where

$$D(\lambda) \neq 0. \tag{9}$$

Divide (8) by $D(\lambda)$ and let

$$\mathbf{K}(x, y; \lambda) = \frac{D\left(\begin{matrix}x\\y\end{matrix}; \lambda\right)}{D(\lambda)}; \tag{10}$$

then

$$\mathbf{K}(x, y; \lambda) - \lambda \int_a^b K(x, \xi)\mathbf{K}(\xi, y; \lambda)d\xi = K(x, y). \tag{11}$$

The function $\mathbf{K}(x, y; \lambda)$ is known as the "*resolving kernel*" or "*resolvent*" of the given kernel $K(x, y)$, because (I) may be solved by its use.

Evidently $\mathbf{K}(x, y; \lambda)$, being the quotient of two integral functions, is a *meromorphic function* of $\lambda$, or has no singularities except poles, in the entire complex $\lambda$-plane.

The equation (11) may be regarded as a special integral equation of the second kind. It is obtained from (I) by placing the kernel $K(x, y)$ itself in place of $f(x)$ (hence another variable $y$ enters as a parameter). The part of unknown function $\phi(x)$ here is played by the resolvent kernel $\mathbf{K}(x, y; \lambda)$.

If the determinant $K\left(\begin{matrix}x, & \xi_1, & \xi_2, & \ldots\ldots, & \xi_n\\ y, & \xi_1, & \xi_2, & \ldots\ldots, & \xi_n\end{matrix}\right)$ be expanded by elements of the first column instead of the first line, and

the same methods be followed through, a relation similar to (4) is obtained; and from this, instead of (8) and (11), the equations

$$(8^*) \qquad D\begin{pmatrix} x \\ y \end{pmatrix}; \lambda\Big) = K(x, y)D(\lambda) + \lambda\int_a^b D\begin{pmatrix} x \\ \xi \end{pmatrix}; \lambda\Big)K(\xi, y)\, d\xi$$

and

$$(11^*) \qquad \boldsymbol{K}(x, y;\ \lambda) - \lambda\int_a^b \boldsymbol{K}(x, \xi;\ \lambda)K(\xi, y)d\xi = K(x, y).$$

Before solving the integral equation (I) by the aid of the resolvent kernel, we will establish the convergence of the series (6) and (7). The convergence proof depends on the *Hadamard theorem on determinants* (see Art. 7).

By this theorem, the determinant

$$A = \begin{vmatrix} a_{11} \ldots\ldots a_{1n} \\ \ldots\ldots\ldots \\ a_{n1} \ldots\ldots a_{nn} \end{vmatrix}$$

satisfies the inequality

$$|A| < k^n\sqrt{n^n}$$

if all the elements satisfy the inequality $|a_{rs}| < k$. The elements may be complex.

Let $M$ be a bound to the absolute value of $K$, so that

$$|K(x, y)| < M$$

for $a \leq x \leq b, a \leq y \leq b$; then from (2), for $n = 1, 2, 3, \ldots,$

$$|C_n| < \frac{1}{n!}M^n\sqrt{n^n}\int_a^b \ldots \int_a^b d\xi_1 \ldots d\xi_n = \frac{M^n(b-a)^n\sqrt{n^n}}{n!}.$$

Hence the series (6) is dominated by the series

$$1 + \sum_{n=1}^{\infty} \frac{M^n(b-a)^n\,|\lambda|^n n^{n/2}}{n!} = \Sigma u_n,$$

so that it is sufficient to prove the convergence of this series. Now

$$\frac{u_{n+1}}{u_n} = \frac{M(b-a)\,|\lambda|}{n+1}\sqrt{\frac{(n+1)^{n+1}}{n^n}} = \frac{M(b-a)\,|\lambda|}{\sqrt{n+1}}\sqrt{\left(1+\frac{1}{n}\right)^n}.$$

But since
$$\left(1+\frac{1}{n}\right)^n \to e = 2.718 +,$$
it follows that $\frac{u_{n+1}}{u_n} \to 0$ for every $\lambda$ and the series in question converges by the ratio test, no matter what value $\lambda$ may have. The proof of the convergence of the series $\sum(-\lambda)^n C_n(x, y)$ is made in the same manner.

We claim now that the integral equation (I) has one and only one solution $\phi(x)$, given by

$$\text{(II)} \qquad \phi(x) = f(x) + \lambda \int_a^b \boldsymbol{K}(x, \xi; \lambda) f(\xi) d\xi,$$

where $\boldsymbol{K}(x, y; \lambda)$ is the resolvent defined by (10).

It is necessary to prove that any solution of (I) must have the form (II), and that conversely the function defined by (II) is always a solution of (I).

First, let $\phi(x)$ be a solution of (I); then multiplying by the resolvent, integrating and interchanging the integration variables, we find

$$\int_a^b \phi(\xi)\boldsymbol{K}(x, \xi; \lambda)d\xi - \lambda \int_a^b \int_a^b K(\xi, y)\boldsymbol{K}(x, \xi; \lambda)\phi(y)dyd\xi$$
$$= \int_a^b f(\xi)\boldsymbol{K}(x, \xi$$

Multiply this by $\lambda$ and add to (I), which gives

$$\begin{aligned} f(x) &+ \lambda \int_a^b \boldsymbol{K}(x, \xi; \lambda) f(\xi) d\xi \\ &= \phi(x) + \lambda \int_a^b \boldsymbol{K}(x, \xi; \lambda)\, \phi(\xi) d\xi - \lambda \int_a^b K(x, \xi)\phi(\xi)d\xi \\ &\quad - \lambda^2 \int_a^b \int_a^b K(\xi, y)\, \boldsymbol{K}(x, \xi; \lambda)\, \phi(y) dy d\xi \\ &= \phi(x) + \lambda \int_a^b \phi(y) dy \left\{ \boldsymbol{K}(x, y; \lambda) - K(x, y) \right. \\ &\qquad\qquad \left. - \lambda \int_a^b \boldsymbol{K}(x, \xi; \lambda)\, K(\xi, y) d\xi \right\} \\ &= \phi(x) \end{aligned}$$

by using (11*).

Conversely, if we substitute (II) in (I), we have

$$
\begin{aligned}
f(x) + \lambda\int_a^b \mathbf{K}(x, \xi; \lambda) f(\xi)\, d\xi - \lambda\int_a^b K(x, \xi) f(\xi)\, d\xi \\
- \lambda^2 \int_a^b\int_a^b K(x, \xi)\, \mathbf{K}(\xi, y; \lambda) f(y)\, d\xi dy \\
= f(x) + \lambda\int_a^b f(y)\, dy \Big\{ \mathbf{K}(x, y; \lambda) - K(x, y) \\
- \lambda \int_a^b K(x, \xi)\, \mathbf{K}(\xi, y; \lambda)\, d\xi \Big\} \\
= f(x)
\end{aligned}
$$

by using (11), which completes the proof.

Hence we have proved the first theorem of Fredholm:

*When $\lambda$ is not a zero of $D(\lambda)$, the integral equation (I) has one and only one solution, which is given by (II).*

It is evident that $\phi(x)$ is continuous, since $f(x)$ and $K(x, y)$ were assumed continuous, and the continuity of the resolvent follows.

By means of an investigation entirely similar to that above, it may be shown that the "associated" integral equation

$$\text{(I*)} \qquad \psi(x) - \lambda \int_a^b K(\xi, x)\psi(\xi) d\xi = g(x),$$

under the hypothesis

$$D(\lambda) \neq 0,$$

has a unique solution given by

$$\text{(II*)} \qquad \psi(x) = g(x) + \lambda \int_a^b \mathbf{K}(\xi, x; \lambda)\, g(\xi) d\xi.$$

It may be easily shown that the associated kernels $K(x, y)$ and $K(y, x)$ have the same determinant $D(\lambda)$ (see also Art. 4).

## Art. 3. The Minor Determinants of D($\lambda$)

In order to be able to consider the treatment of the case $D(\lambda) = 0$, it is necessary to obtain, in addition to $D(\lambda)$ and

$D\binom{x}{y}; \lambda)$, certain other entire functions of $\lambda$ which are called "minors of higher order" of $D(\lambda)$, as well as the relations between these and $D(\lambda)$.

From the definitions (2) and (3) above,

$$nC_n = \int_a^b C_{n-1}(\xi, \xi)d\xi,$$

so that

$$\sum_{n=0}^{\infty} n(-\lambda)^{n-1}C_n = \sum_{n=0}^{\infty} \int_a^b (-\lambda)^{n-1} C_{n-1}(\xi,\xi)\, d\xi$$

or

$$\frac{dD(\lambda)}{d\lambda} = -\int_a^b D\left(\begin{matrix}\xi\\ \xi\end{matrix}; \lambda\right) d\xi. \tag{12}$$

In order to define the minors of higher order, let[5]

$$\left\{\begin{aligned} &C_n\begin{pmatrix}x_1, x_2, \ldots, x_m\\ y_1, y_2, \ldots, y_m\end{pmatrix} = \frac{1}{n!}\int_a^b \cdots \int_a^b \\ &\qquad\qquad K\begin{pmatrix}x_1, x_2, \ldots, x_m, \xi_1, \xi_2, \ldots, \xi_n\\ y_1, y_2, \ldots, y_m, \xi_1, \xi_2, \ldots, \xi_n\end{pmatrix} d\xi_1 \ldots d\xi_n, \\ &C_0\begin{pmatrix}x_1, x_2, \ldots, x_m\\ y_1, y_2, \ldots, y_m\end{pmatrix} = K\begin{pmatrix}x_1, x_2, \ldots, x_m\\ y_1, y_2, \ldots, y_m\end{pmatrix}.\end{aligned}\right. \tag{13}$$

Then

$$D\left(\begin{matrix}x_1, x_2, \ldots, x_m\\ y_1, y_2, \ldots, y_m\end{matrix}; \lambda\right) = \sum_{n=0}^{\infty} (-\lambda)^n C_n\begin{pmatrix}x_1, x_2, \ldots, x_m\\ y_1, y_2, \ldots, y_m\end{pmatrix} \tag{14}$$

is *the minor of the m-th order*, $m = 1, 2, 3, \ldots$.

By the aid of the Hadamard theorem, it can be proved that the series (14) converges for all $\lambda$. Moreover,

$$\frac{d^m D(\lambda)}{d\lambda^m} = (-1)^m \int_a^b \cdots \int_a^b D\left(\begin{matrix}\xi_1, \xi_2, \ldots, \xi_m\\ \xi_1, \xi_2, \ldots, \xi_m\end{matrix}; \lambda\right) d\xi_1 \ldots d\xi_m \tag{15}$$

which contains (12) as a special case for $m = 1$.

---

[5]The above functions $C_n(x, y)$ are now to be designated by $C_n\binom{x}{y}$.

By developing the determinant

$$K\begin{pmatrix} x_1, x_2, \ldots, x_m, \xi_1, \ldots, \xi_n \\ y_1, y_2, \ldots, y_m, \xi_1, \ldots, \xi_n \end{pmatrix}$$

by the elements of the first line, we get the following generalization of (4),

$$\begin{aligned} &K\begin{pmatrix} x_1, \ldots, x_m, \xi_1, \ldots, \xi_n \\ y_1, \ldots, y_m, \xi_1, \ldots, \xi_n \end{pmatrix} \\ &= K(x_1,y_1)\, K\begin{pmatrix} x_2, \ldots, x_m, \xi_1, \ldots, \xi_n \\ y_2, \ldots, y_m, \xi_1, \ldots, \xi_n \end{pmatrix} + \cdots \\ &+(-1)^{m-1}K(x_1,y_m)K\begin{pmatrix} x_2, \ldots, x_m, \xi_1, \ldots, \xi_n \\ y_1, \ldots, y_{m-1}, \xi_1, \ldots, \xi_n \end{pmatrix} \\ &+(-1)^m K(x_1,\xi_1)\, K\begin{pmatrix} x_2, \ldots, x_m, \xi_1, \xi_2, \ldots, \xi_n \\ y_1, \ldots, y_{m-1}, y_m, \xi_2, \ldots \xi_n \end{pmatrix} + \cdots \end{aligned} \tag{16}$$

Division by $n!$ and integration with respect to $\xi_1, \xi_2, \ldots, \xi_n$ gives the identity

$$\begin{aligned} C_n\begin{pmatrix} x_1, \ldots, x_m \\ y_1, \ldots, y_m \end{pmatrix} &= K(x_1,y_1)\, C_n\begin{pmatrix} x_2, \ldots, x_m \\ y_2, \ldots, y_m \end{pmatrix} + \cdots \\ &+ (-1)^{m-1}K(x_1,y_m)\, C_n\begin{pmatrix} x_2, \ldots, x_m \\ y_1, \ldots, y_{m-1} \end{pmatrix} \\ &- \int_a^b K(x_1,\xi)\, C_{n-1}\begin{pmatrix} \xi, x_2, \ldots, x_m \\ y_1, y_2, \ldots, y_m \end{pmatrix} d\xi \end{aligned} \tag{17}$$

for $n = 1, 2, 3, \ldots$. However, since $C_{-1}\begin{pmatrix} \xi, x_2, \ldots, x_m \\ y_1, y_2, \ldots, y_m \end{pmatrix} \equiv 0$, this identity is also valid for $n = 0$, when it gives the expansion by elements of the first line of $K\begin{pmatrix} x_1, \ldots, x_m \\ y_1, \ldots, y_m \end{pmatrix}$.

Multiplying the equations (17) by $(-\lambda)^n$ and summing, we find the generalization of (8),

$$
\begin{aligned}
D\begin{pmatrix} x_1, x_2, \ldots, x_m \\ y_1, y_2, \ldots, y_m \end{pmatrix}; \lambda\Big) \\
&= K(x_1, y_1)\, D\begin{pmatrix} x_2, \ldots, x_m \\ y_2, \ldots, y_m \end{pmatrix}; \lambda\Big) \\
&\quad \vdots \\
&\quad + (-1)^{m-1} K(x_1, y_m)\, D\begin{pmatrix} x_2, \ldots, x_m \\ y_1, \ldots, y_{m-1} \end{pmatrix}; \lambda\Big) \\
&\quad + \lambda \int_a^b K(x_1, \xi)\, D\begin{pmatrix} \xi, x_2, \ldots, x_m \\ y_1, y_2, \ldots, y_m \end{pmatrix}; \lambda\Big)\, d\xi.
\end{aligned} \tag{18}
$$

## Art. 4. The Second Theorem of Fredholm

From the first Fredholm theorem, it is evident that the homogeneous equation

$$\phi(x) - \lambda \int_a^b K(x, \xi)\phi(\xi)d\xi = 0, \tag{19}$$

in which $f(x) \equiv 0$, has no solution except the trivial one $\phi(x) \equiv 0$, when $D(\lambda) \neq 0$.

We now make the hypothesis that

$$D(\lambda) = 0, \tag{20}$$

and will prove that in this case (19) possesses non-trivial solutions.

First let $c$ be a *simple* zero of $D(\lambda)$. Then $D\begin{pmatrix} x \\ y \end{pmatrix}; c\Big)$ cannot vanish identically in $x$ and $y$; for then we would have $D\begin{pmatrix} \xi \\ \xi \end{pmatrix}; c\Big) = 0$ identically in $\xi$, so that from (12), $D'(c) = 0$, which is impossible when $c$ is a simple zero of $D(\lambda)$.

Since $D(c) = 0$, equation (8) becomes

$$D\begin{pmatrix} x \\ y \end{pmatrix}; c\Big) - c\int_a^b K(x, \xi) D\begin{pmatrix} \xi \\ y \end{pmatrix}; c\Big) d\xi = 0. \tag{21}$$

Choose for $(x, y)$ a number pair $(x_1, y_1)$ such that $D\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}; c\Big) \neq 0$,

which is possible since $D\begin{pmatrix} x \\ y \end{pmatrix}; c\Big)$ is not identically zero. Then (21) shows that $D\begin{pmatrix} x & ; c \\ y_1 & \end{pmatrix}$ with fixed $y_1$ is a solution of (19) which does not vanish identically.

If $D\begin{pmatrix} x & ; c \\ y & \end{pmatrix}$ vanishes identically, then from (12) $D'(c) = 0$; but the converse of this is not true. We will now make, instead of the assumption that $c$ is a simple zero of $D(\lambda)$, the more general hypothesis that $D(c) = 0$ while $D\begin{pmatrix} x & ; c \\ y & \end{pmatrix}$ is not identically zero; then we find as before that $\phi(x) = D\begin{pmatrix} x & ; c \\ y_1 & \end{pmatrix}$ is a solution of (19). In this case, $c$ is *at least* a simple zero of $D(\lambda)$.

We will now generalize the discussion, by assuming that for $\lambda = c$, not merely $D(\lambda)$, but also all the minor determinants to order $m - 1$ vanish identically in their variables, while the $m$-th order minor $D\begin{pmatrix} x_1, x_2, \ldots\ldots, x_m \\ y_1, y_2, \ldots\ldots, y_m \end{pmatrix}$ does not vanish identically. From (15) it follows that $D'(c), \ldots\ldots, D^{(m-1)}(c)$ are zero, or $c$ is at least an $m$-fold zero of $D(\lambda)$.

The identity (18) then becomes

$$
(22) \quad D\begin{pmatrix} x_1, \ldots., x_m & ; c \\ y_1, \ldots., y_m & \end{pmatrix} - c\int_a^b K(x_1, \xi) D\begin{pmatrix} \xi, & x_2, \ldots. x_m & ; c \\ y_1, & y_2, \ldots. y_m & \end{pmatrix} d\xi = 0,
$$

or, writing $x$ in place of $x_1$,

$$
(22^*) \quad D\begin{pmatrix} x, x_2, \ldots., x_m & ; c \\ y_1, y_2, \ldots., y_m & \end{pmatrix} - c\int_a^b K(x, \xi) D\begin{pmatrix} \xi, x_2, \ldots, x_m & ; c \\ y_1, \ldots\ldots, y_m & \end{pmatrix} d\xi = 0.
$$

Hence $D\begin{pmatrix} x, x_2, \ldots., x_m & ; c \\ y_1, \ldots\ldots., y_m & \end{pmatrix}$ is a non-trivial solution of (19), in

case $x_2, \ldots, x_m, y_1, \ldots, y_m$ are given such values that the function is not identically zero.

In a similar manner, we find in general $m$ solutions

$$D\begin{pmatrix} x_1, \ldots, x_{k-1}, x, x_{k+1}, \ldots, x_m \\ y_1, \ldots\ldots\ldots\ldots\ldots, y_m \end{pmatrix}; c\Big),$$

for $k = 1, 2, \ldots, m$, since each of the number-pairs $\begin{pmatrix} x_k \\ y_k \end{pmatrix}$ can be put in the first place in the expansion of the determinant. Following Fredholm, we write these solutions in the form

$$(23) \quad \phi_k(x) = \frac{D\begin{pmatrix} x_1, \ldots, x_{k-1}, x, x_{k+1}, \ldots, x_m \\ y_1, \ldots\ldots\ldots\ldots\ldots, y_m \end{pmatrix}; c\Big)}{D\begin{pmatrix} x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_m \\ y_1, \ldots\ldots\ldots\ldots\ldots, y_m \end{pmatrix}; c\Big)}, \; k = 1, 2, \ldots, m.$$

The denominator is independent of $x$ and hence constant, and from any solution of (19) others may be obtained by multiplying or dividing by a constant.

Evidently

$$\phi_k(x_k) = 1, \text{ for } k = 1, 2, \ldots, m.$$

We will next prove that the $m$ solutions are linearly independent. We note that by using (23) and the relation $\phi_1(x_1) = 1$, (22) can be written

$$(22^{**}) \qquad 1 - c\int_a^b K(x_1, \xi)\phi_1(\xi)d\xi = 0.$$

In general,

$$(24) \qquad \int_a^b K(x_k, \xi)\phi_k(\xi)d\xi = \frac{1}{c}, \; k = 1, 2, \ldots, m.$$

In (16), let $x_1 = x_2$, so that the determinant on the left takes the form

$$K\begin{pmatrix} x_2, x_2, x_3, \ldots x_m, \xi_1, \ldots \xi_n \\ y_1, y_2, y_3, \ldots y_m, \xi_1, \ldots \xi_n \end{pmatrix}$$

and accordingly vanishes identically, since the determinant has two lines alike. By expansions which are analogous to those above, we get instead of (22**) the equation

$$\int_a^b K(x_2, \xi)\phi_1(\xi)d\xi = 0.$$

In general,

$$\int_a^b K(x_r, \xi)\phi_s(\xi)d\xi = 0, \; r \neq s. \tag{25}$$

Now if $\phi_1, \phi_2, \ldots$ were linearly dependent, that is, if there existed a relation

$$c_1\phi_1(x) + c_2\phi_2(x) + \ldots + c_m\phi_m(x) \equiv 0$$

with constant coefficients, then it would follow that for $k = 1, 2, \ldots, m$,

$$c_1 \int^b K(x_k, \xi)\phi_1(\xi)d\xi + \ldots\ldots + c_m \int^b K(x_k, \xi)\phi_m(\xi)d\xi = 0,$$

and hence on account of (24) and (25),

$$c_k = 0, \; k = 1, 2, \ldots, m;$$

this completes the proof that the functions (23) are linearly independent.

Finally, we will show that *every* solution of (19) is a linear combination with constant coefficients of the functions $\phi_1(x), \ldots, \phi_m(x)$. (The converse theorem that such a linear combination is a solution is trivial, of course, in consequence of the linear homogeneous character of the equation.)

Let $\phi(x)$ be any solution of (I) (later we will set $f(x) \equiv 0$); then

$$\int_a^b L(x, \xi)\phi(\xi)d\xi - \lambda \int_a^b\int_a^b L(x, \xi)K(\xi, y)\phi(y)d\xi dy = \int_a^b L(x, \xi)f(\xi)d\xi$$

where $L(x, \xi)$ is any continuous function of $x$ and $\xi$. Multiply this equation by $\lambda$ and add it to (I), obtaining

$$(26)\quad \phi(x) - \lambda \int_a^b G(x, \xi)\phi(\xi)d\xi = f(x) + \lambda \int_a^b L(x, \xi)f(\xi)d\xi,$$

an integral equation with the kernel

$$(27)\quad G(x, \xi) = K(x, \xi) - L(x, \xi) + \lambda \int_a^b L(x, y)K(y, \xi)dy.$$

(This equation was used in Art. 2 with $L(x, \xi) = \mathbf{K}(x, \xi; \lambda)$ and $G(x, \xi) \equiv 0$). We now let

$$(28)\qquad L(x, \xi) = \frac{D\begin{pmatrix} x, \xi_1, \ldots, \xi_n \\ \xi, \eta_1, \ldots, \eta_n \end{pmatrix}; c}{D\begin{pmatrix} \xi_1, \ldots, \xi_n \\ \eta_1, \ldots, \eta_n \end{pmatrix}; c}.$$

Then by expanding

$$D\left(\begin{matrix} x, \xi_1, \ldots, \xi_n \\ \xi, \eta_1, \ldots, \eta_n \end{matrix}; c\right)$$

similarly to (18), we get the equation

$$(29)\qquad G(x, \xi) = \frac{1}{D\left(\begin{matrix} \xi_1, \xi_2, \ldots, \xi_n \\ \eta_1, \ldots \eta_n \end{matrix}; c\right)} \left\{ \begin{aligned} &K(\xi_1, \xi)\, D\left(\begin{matrix} x, \xi_2, \ldots, \xi_n \\ \eta_1, \ldots \eta_n \end{matrix}; c\right) \\ &\quad + \ldots \\ &+ (-1)^n K(\xi_n, \xi) D\left(\begin{matrix} \xi_1, \ldots, \xi_{n-1}, x \\ \eta_1, \ldots, \eta_n \end{matrix}; c\right) \end{aligned} \right\}.$$

Remembering that every solution of (I) must satisfy (26), and hence every solution of (19) must satisfy

$$\phi(x) = \lambda \int_a^b G(x, \xi)\phi(\xi)d\xi,$$

we insert the expression (29) for $G$, and noting the equations (23), we see that we have obtained $\phi(x)$ as a linear combination of the $\phi_k(x)$.

On investigating the equation

$$\psi(x) - \lambda\int_a^b K(\xi, x)\psi(\xi)d\xi = 0, \tag{19*}$$

associated with (19), under the same hypotheses, it is found, that this likewise has $m$ linearly independent solutions

$$\psi_k(x) = \frac{D\begin{pmatrix} y_1, \ldots\ldots\ldots\ldots\ldots\ldots, y_m \\ x_1, \ldots, x_{k-1}, x, x_{k+1}, \ldots, x_m \end{pmatrix}; c}{D\begin{pmatrix} y_1, \ldots\ldots\ldots\ldots\ldots\ldots, y_m \\ x_1, \ldots\ldots\ldots\ldots\ldots\ldots, x_m \end{pmatrix}; c}, \tag{30}$$

and that the general solution is a linear combination of these.

For if we designate the quantities belonging to the associated kernel $K(\xi, x)$ by bars, then we have obviously

$$\overline{K}\begin{pmatrix} x_1, \ldots, x_n \\ y_1, \ldots, y_n \end{pmatrix} = K\begin{pmatrix} y_1, \ldots, y_n \\ x_1, \ldots, x_n \end{pmatrix},$$

and hence

$$\overline{C}_n = C_n, \ \overline{C}_n(x, y) = C_n(y, x);$$

therefore

$$\overline{D}(\lambda) = D(\lambda), \ \overline{D}\begin{pmatrix} x \\ y \end{pmatrix}; \lambda\Big) = D\begin{pmatrix} y \\ x \end{pmatrix}; \lambda\Big),$$

and in general

$$\overline{D}\begin{pmatrix} x_1, \ldots, x_m \\ y_1, \ldots, y_m \end{pmatrix}; \lambda\Big) = D\begin{pmatrix} y_1, \ldots, y_m \\ x_1, \ldots, x_m \end{pmatrix}; \lambda\Big),$$

from which the solutions (30) follow.

We will collect the results of this article into the following theorem:

*If $\lambda = c$ is a zero of the determinant $D(\lambda)$, and if the minor determinants of orders $1, 2, \ldots m-1$ vanish identically for $\lambda = c$, while that of order $m$ does not, then the homogeneous equations*

$$\phi(x) - c\int_a^b K(x, \xi)\phi(\xi)d\xi = 0,$$

$$\psi(x) - c\int_a^b K(\xi, x)\psi(\xi)d\xi = 0,$$

*each possess exactly $m$ linearly independent solutions $\phi_k(x)$ and*

*$\psi_k(x)$ which are given by (23) or (30) respectively. The most general solutions of the above equations are linear combinations of these functions $\phi_k(x)$ or $\psi_k(x)$.*

In particular, we have the *second Fredholm theorem*:

*If $\lambda = c$ is a root of the m-th order of the equation $D(\lambda) = 0$, then the homogeneous equation*

$$\phi(x) - c\int_a^b K(x, \xi)\phi(\xi)d\xi = 0$$

*has at least one and at most m linear independent solutions.*

The roots of the equation $D(\lambda) = 0$ are called *eigen-values*, and the corresponding solutions of the homogeneous integral equation are the *eigen-functions* for the kernel $K(x, \xi)$. Any root $c$ of $D(\lambda) = 0$ can have only a finite multiplicity, since $D(\lambda)$ is an integral function. Accordingly, there can only be a finite number of eigen-functions corresponding to any one eigen-value. Moreover, the kernel may not have any eigen-values. Only for symmetric kernels (for which $K(x, y) \equiv K(y, x)$) has it been proved in general that the equation $D(\lambda) = 0$ must have atleast one root; however, we cannot go into this subject any deeper.

## Art. 5. The Third Theorem of Fredholm

We will now investigate again the non-homogeneous equation

$$\phi(x) - c\int_a^b K(x, \xi)\phi(\xi)d\xi = f(x) \tag{31}$$

under the assumption that $c$ is a zero of $D(\lambda)$, and that the minor determinants of orders $1, 2, \ldots m - 1$ vanish identically for $\lambda = c$, but that the minor of order $m$ does not vanish identically.

We will show first that (31) in general has no solution, that is, for arbitrary choice of the function $f(x)$; and moreover,

that certain conditions on $f(x)$ are necessary and sufficient for the existence of a solution.

First, let $\phi(x)$ be a solution of (31); then on multiplying with $\psi_k(x)$ (see (30)) and integrating, we find

$$\int_a^b f(x)\psi_k(x)dx = \int_a^b \phi(x)\psi_k(x)\,dx - c\iint K(x,\xi)\phi(\xi)\psi_k(x)dxd\xi$$
$$= \int_a^b \phi(x)dx\left\{\psi_k(x) - c\int_a^b K(\xi,x)\psi_k(\xi)d\xi\right\} = 0$$

from equation (19*).

Hence the $m$ conditions

$$(32) \qquad \int_a^b f(x)\psi_k(x)dx = 0, \quad (k = 1, 2, \ldots, m)$$

are necessary conditions which $f(x)$ must satisfy if (31) is to have a solution. That is, $f(x)$ must be orthogonal to each of the $\psi_k(x)$.

These $m$ conditions are also sufficient, for if they are satisfied then

$$(33) \quad \Phi(x) = f(x) + \frac{c}{D\begin{pmatrix}\xi_1, \ldots, \xi_m \\ \eta_1, \ldots, \eta_m\end{pmatrix}; c} \int_a^b D\begin{pmatrix}x, \xi_1, \ldots, \xi_m \\ y, \eta_1, \ldots, \eta_m\end{pmatrix}; c\Big) f(y)dy$$

is a solution of (31).

In order to see this, substitute this function for $\phi$ in (31), which gives, after a simple reduction,

$$\frac{1}{D\left(\begin{matrix}\xi_1, \ldots, \xi_m \\ \eta_1, \ldots, \eta_m\end{matrix}; c\right)} \int_a^b D\left(\begin{matrix}x, \xi_1, \ldots, \xi_m \\ y, \eta_1, \ldots, \eta_m\end{matrix}; c\right) f(y)dy$$
$$= \int_a^b K(x,\xi)\left\{f(\xi) + \frac{c}{D\left(\begin{matrix}\xi_1, \ldots, \xi_m \\ \eta_1, \ldots, \eta_m\end{matrix}; c\right)} \int_a^b D\left(\begin{matrix}\xi, \xi_1, \ldots, \xi_m \\ y, \eta_1, \ldots, \eta_m\end{matrix}; c\right) f(y)\,dy\right\} d\xi.$$

This equation leads, if we expand as in (18),

$$D\begin{pmatrix} x, \xi_1, \ldots, \xi_m \\ y, \eta_1, \ldots, \eta_m \end{pmatrix}; c\Big) = K(x, y)\; D\begin{pmatrix} \xi_1, \ldots, \xi_m \\ \eta_1, \ldots, \eta_m \end{pmatrix}; c\Big)$$

$$- K(x, \eta_1)\, D\begin{pmatrix} \xi_1, \ldots\ldots, \xi_m \\ y, \eta_2, \ldots, \eta_m \end{pmatrix}; c\Big)$$

$$\vdots$$

$$+(-1)^m K(x, \eta_m)\, D\begin{pmatrix} \xi_1, \ldots\ldots \xi_m \\ y, \eta_1, \ldots, \eta_{m-1} \end{pmatrix}; c\Big)$$

$$+ c\int_a^b K(x, \xi)\, D\begin{pmatrix} \xi, \xi_1, \ldots, \xi_m \\ y, \eta_1, \ldots, \eta_m \end{pmatrix}; c\Big)\, d\xi,$$

and use the definition equations (30), to an equation of the form

$$\int_a^b f(y)\left\{a_1\psi_1(y) + \ldots\ldots + a_m\psi_m(y)\right\} dy = 0,$$

where the $a_k$ are free of $y$. But this equation is satisfied, because $f$ is orthogonal to all the $\psi_k$, and hence $\Phi$ is a solution of (31).

Having found that a solution exists under the hypotheses (32), we must note that this is not the only solution. On account of the linear character of the equation (31), we can add any linear combination of solutions of the corresponding homogeneous equation, and still have a solution. Hence the most general solution of (31) is of the form

$$\phi(x) = \Phi(x) + \sum_{k=1}^{m} c_k\phi_k(x). \tag{34}$$

We can now state the *third Fredholm theorem*:

*If $c$ is a root of the equation $D(\lambda) = 0$, then for the existence of a solution of the non-homogeneous equation* (31) *it is necessary and sufficient that $f(x)$ be orthogonal to all the solutions $\psi_k(x)$ of the associated homogeneous equation; if these conditions are*

*fulfilled, then the general solution of* (31) *is found by adding to the solution* $\Phi$ *defined by* (33) *a linear combination of the solutions of the corresponding homogeneous equation.*

It is evident that the Fredholm theorems on linear integral equations of the second kind show a very complete analogy with the theory of linear equations in algebra, when the number of algebraic equations is equal to the number of unknowns.

The Fredholm theorems lead immediately to the following theorem, which we will use in the solution of boundary value problems:

*If the homogeneous equation*

$$\phi(x) - \lambda \int_a^b K(x, \xi)\phi(\xi)d\xi = 0$$

*has no solution, then the non-homogeneous equation*

$$\phi(x) - \lambda \int_a^b K(x, \xi)\phi(\xi)d\xi = f(x)$$

*has a unique solution.*

*If the homogeneous equation has exactly m linearly independent solutions* $\phi_1, \ldots \phi_m$, *then the associated homogeneous equation has likewise m linearly independent solutions* $\psi_1, \ldots, \psi_m$. *The non-homogeneous equation in this case has a solution if and only if* $f(x)$ *is orthogonal to all the solutions* $\psi_k$, *and if this condition is satisfied, the general solution contains m arbitrary constants.*

For if the homogeneous equation has no non-trivial solution, then $D(\lambda) \neq 0$, from which the first part of this theorem follows. On the other hand, if the homogeneous equation has exactly $m$ independent solutions, then $D(\lambda)$ and all the minor determinants down to the order $m - 1$ vanish identically, but not the one of order $m$, from which the second part of the theorem follows. (We have used here the converse of the theorem on page 275, which is obviously true also.)

### Art. 6. The Iterated Kernels

We found in Art. 2 that the resolvent,

$$\mathbf{K}(x, y ; \lambda) = \frac{D\begin{pmatrix} x \\ y \end{pmatrix}; \lambda)}{D(\lambda)},$$

is an analytic function of the complex parameter $\lambda$, and indeed is a meromorphic function. Since the denominator $D(\lambda)$ does not vanish for $\lambda = 0$ (for we have $D(0) = 1$), the resolvent is regular near the origin and can be expanded in a power series

$$\mathbf{K}(x, y ; \lambda) = \sum_{n=0}^{\infty} K^{n+1}(x, y)\, \lambda^n \tag{35}$$

in the neighbourhood of $\lambda = 0$. This series converges in the maximum circle which contains no singular point of $\mathbf{K}$, that is, in the largest circle which contains no zero of $D(\lambda)$. Hence, if $c_0$ is the eigen-value of smallest absolute value, then $|c_0|$ is the radius of convergence. If the kernel $K(x, y)$ has no eigen-values, then the resolvent $\mathbf{K}(x, y; \lambda)$ is itself an entire function.

The coefficients of the power series, which have been designated by $K^n(x, y)$, may be found by recursion formulas obtained by substituting (35) in (11) and equating coefficients of like powers of $\lambda$. For the first function we have

$$K^1(x, y) = \mathbf{K}(x, y ; 0) = K(x, y). \tag{36}$$

The succeeding functions are

$$K^2(x, y) = \int_a^b K(x, \xi) K(\xi, y) d\xi,$$

$$K^3(x, y) = \int_a^b K(x, \xi) K^2(\xi, y) d\xi,$$

$$\cdots\cdots\cdots$$

and in general

$$K^{n+1}(x, y) = \int^b K(x, \xi) K^n(\xi, y) d\xi, \quad (n = 1, 2, \ldots). \tag{37}$$

It is easily shown that

$$K^{n+1}(x, y) = \int_a^b \cdots \int_a^b K(x, \xi_1)\, K(\xi_1, \xi_2) \ldots\ldots K(\xi_n, y) d\xi_1 \ldots d\xi_n,$$

$$(37^*) \qquad K^{m+n}(x, y) = \int_a^b K^m(x, \xi) K^n(\xi, y) d\xi,$$

and in particular

$$K^{2n}(x, y) = \int_a^b K^n(x, \xi) K^n(\xi, y)\, d\xi.$$

The functions $K^n(x, y)$ are called the *iterated kernels.* They are of importance in the extension of the Fredholm theory to non-bounded kernels (see Art. 8).

## Art. 7. Proof of Hadamard's Theorem

Given the determinant

$$(38) \qquad A = \begin{vmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ \ldots & \ldots & \ldots\ldots & \ldots \\ a_{n1} & a_{n2} & \ldots\ldots & a_{nn} \end{vmatrix} = [a_{rs}],$$

let

$$(39) \qquad c_r = \sum_{i=1}^{n} |a_{ri}|^2 = |a_{r1}|^2 + |a_{r2}|^2 + \ldots + |a_{rn}|^2.$$

Then Hadamard's theorem is that

$$(40) \qquad |A|^2 \leq c_1 c_2 \ldots c_n.$$

To prove this, introduce the conjugate determinant $\bar{A}$ whose elements $\bar{a}_{rs}$ are the complex conjugates of the corresponding elements of $A$, and the quantities

$$b_{rs} = \frac{a_{rs}}{\sqrt{c_r}}, \qquad \bar{b}_{rs} = \frac{\bar{a}_{rs}}{\sqrt{c_r}}, \qquad (r, s = 1, 2, \ldots, n)$$

and their determinants

$$B = [b_{rs}], \quad \bar{B} = [\bar{b}_{rs}].$$

Then we have to prove that

$$(40^*) \qquad |B|^2 = B\bar{B} \leq 1.$$

From (39) we obtain

$$\sum_{i=1}^{n} b_{ri}\bar{b}_{ri} = 1, \quad (r = 1, 2, \ldots, n). \tag{39*}$$

Let us find the maximum value of $B\bar{B}$, as a function of $b_{rs}$, $\bar{b}_{rs}$, subject to the conditions (39*). We express this in terms of real variables by writing

$$b_{rs} = x_{rs} + iy_{rs},$$
$$\bar{b}_{rs} = x_{rs} - iy_{rs}.$$

Then the function $B\bar{B}$ is to be made a maximum as a function of the real variables $x_{rs}$, $y_{rs}$, which are not independent but subject to the conditions (39*). By the Lagrange multiplier method, this is equivalent to making the function

$$B\bar{B} - l_1 \sum_{s=1}^{n} b_{1s}\bar{b}_{1s} \ldots - l_n \sum_{s=1}^{n} b_{ns}\bar{b}_{ns}$$

a maximum, where the Lagrange multipliers $l_r$ are constants and the variables $x_{rs}$, $y_{rs}$ are treated as if they were all independent. Hence we set each partial derivative equal to zero, and since

$$\frac{\partial B}{\partial x_{rs}} = B_{rs}, \qquad \frac{\partial \bar{B}}{\partial x_{rs}} = \bar{B}_{rs},$$

and

$$\frac{\partial B}{\partial y_{rs}} = iB_{rs}, \qquad \frac{\partial \bar{B}}{\partial y_{rs}} = -i\bar{B}_{rs},$$

where $B_{rs}$ and $\bar{B}_{rs}$ are the cofactors of $b_{rs}$ and $\bar{b}_{rs}$ in $B$ and $\bar{B}$, we find

$$B\bar{B}_{rs} + \bar{B}B_{rs} = 2l_r x_{rs},$$

$$-iB\bar{B}_{rs} + i\bar{B}B_{rs} = 2l_r y_{rs}, \quad (r, s = 1, 2, \ldots n).$$

From these, we find

$$\bar{B}B_{rs} = l_r(x_{rs} - iy_{rs}) = l_r\bar{b}_{rs}.$$

Multiplying by $b_{rs}$ and summing, this gives

$$\bar{B}\sum_{s=1}^{n} B_{rs}b_{rs} = l_r \sum_{s=1}^{n} b_{rs}\bar{b}_{rs},$$

or from (39*)

$$\bar{B}B = l_r.$$

Hence the Lagrange multipliers are all equal, and

$$\bar{b}_{rs} = \frac{\bar{B}B_{rs}}{l_r} = \frac{B_{rs}}{B}.$$

By the rule for multiplication of determinants,

$$B\bar{B} = \left[\sum_{s=1}^{n} b_{rs}\bar{b}_{ts}\right] = \left[\frac{\sum b_{rs}B_{ts}}{B}\right] = \begin{vmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ . & . & . & . \\ 0 & 0 & \ldots & 1 \end{vmatrix} = 1.$$

Hence the maximum value of $B\bar{B}$ is 1, which establishes (40*) and hence the inequality (40). (The minimum value of $B\bar{B}$ is 0.)

If the elements of the determinant $A$ all satisfy the condition

$$|a_{rs}| < k,$$

then from (39), $c_r < nk^2$, and hence from (40), we get the inequality used in Art. 2,

$$|A|^2 < n^n k^{2n}.$$

## Art. 8. Kernels which are not Bounded

The Fredholm theorems remain unchanged when the kernels are not required to be continuous, but are merely supposed bounded and integrable. No change is needed anywhere in the proofs. The situation is different when the kernel $K(x, y)$ is supposed integrable but not bounded; for in this case the auxiliary functions $C_n$, $C_n(x, y)$ may lose their meaning (see below).

But in the solution of boundary value problems we will need to use precisely this case where the kernel is *integrable* though not bounded. Hence we need to investigate under what hypotheses the Fredholm theorems are still valid for unbounded kernels. *We will assume that the n-th iterated kernel $K^n(x, y)$ remains bounded, for fixed n.* Then all the higher iterated kernels $K^{n+1}$, etc., are bounded. The Fredholm theorems are valid for an integral equation with kernel $K^n$. The determinant and the minor determinants for this kernel can be formed, and will be designated by $D_n(\lambda)$ and $D_n\begin{pmatrix} x \\ y \end{pmatrix}; \lambda)$; they are entire functions of $\lambda$.

The resolvent of $K^n$ is

$$(41) \qquad \mathbf{K}_n(x, y; \lambda) = \frac{D_n\left(\begin{matrix} x \\ y \end{matrix}; \lambda\right)}{D_n(\lambda)},$$

or

$$(42) \quad \mathbf{K}_n(x, y; \lambda) = K^n(x, y) + \lambda K^{2n}(x, y) + \dots + \lambda^s K^{(s+1)n}(x, y) + \dots,$$

where $K^n$, $K^{2n}$, ... are iterated kernels. The last form is valid only in a certain neighbourhood of $\lambda = 0$, while the first is valid over the entire $\lambda$-plane.

The resolvent $\mathbf{K}(x, y; \lambda)$ of the original kernel $K(x, y)$ is at first not defined, since $D(\lambda)$ and $D\left(\begin{matrix} x \\ y \end{matrix}; \lambda\right)$ are not defined for an unbounded kernel. In the applications, it is generally the case that $K(x, y)$ becomes infinite for $y = x$, or in other words on the diagonal passing through the origin of the square $a \leq x \leq b$, $a \leq y \leq b$; then the coefficients $C_n$ and $C_n(x, y)$ defined by (2) and (3) lose their meaning, because their definitions contain the term $K(\xi, \xi)$. In order to obtain a definition of the resolvent kernel $\mathbf{K}(x, y; \lambda)$ for a non-bounded kernel, we will first obtain another form of the resolvent kernel $\mathbf{K}$ for a

bounded kernel $K$, a form which will retain its meaning when $K$ is non-bounded but integrable and $K^n$ is bounded.

From (42), for $s = 1, 2, \ldots, n-1$,

$$\int_a^b K^s(x, \xi)\boldsymbol{K}_n(\xi, y; \lambda)d\xi = K^{n+s}(x, y) + \lambda K^{2n+s}(x, y) + \lambda^2 K^{3n+s}(x, y) + \ldots;$$

hence, combining this with

$$\boldsymbol{K}(x, y; \lambda) = K(x, y) + \lambda K^2(x, y) + \lambda^2 K^3(x, y) + \ldots$$

we find

$$\begin{aligned}\boldsymbol{K}(x, y; \lambda) = K(x, y) &+ \lambda K^2(x, y) + \ldots \\ &+ \lambda^{n-2}K^{n-1}(x, y) + \lambda^{n-1}\boldsymbol{K}_n(x, y; \lambda^n) \\ &+ \lambda^n \int_a^b \{K(x, \xi) + \lambda K^2(x, \xi) + \ldots \\ &+ \lambda^{n-2}K^{n-1}(x, \xi)\}\boldsymbol{K}_n(\xi, y; \lambda^n)\, d\xi.\end{aligned}$$

For brevity, let

$$S(x, y; \lambda) = K(x, y) + \lambda K^2(x, y) + \ldots + \lambda^{n-2}K^{n-1}(x, y); \tag{43}$$

then

$$\boldsymbol{K}(x, y; \lambda) = S(x, y; \lambda) + \lambda^{n-1}\boldsymbol{K}_n(x, y; \lambda^n) + \lambda^n \int_a^b S(x, \xi; \lambda)\boldsymbol{K}_n(\xi, y; \lambda^n)\, d\xi. \tag{44}$$

Now this representation of the resolvent $\boldsymbol{K}(x, y; \lambda)$ for bounded kernels $K(x, y)$ retains a meaning in our more general case of an unbounded kernel. Inserting in (44) for $\boldsymbol{K}_n(x, y, \lambda^n)$ the expression (42), where $\lambda^n$ is to replace $\lambda$, we obtain a formula valid in a certain neighbourhood of $\lambda = 0$. But the use of (41) leads to

$$\boldsymbol{K}(x, y; \lambda) = S(x, y; \lambda) + \lambda^{n-1}\frac{D_n\begin{pmatrix} x \\ y \end{pmatrix}; \lambda^n\Big)}{D_n(\lambda^n)} + \lambda^n\int_a^b S(x, \xi; \lambda)\frac{D_n\begin{pmatrix} \xi \\ y \end{pmatrix}; \lambda^n\Big)}{D_n(\lambda^n)}\, d\xi, \tag{44*}$$

and this representation is valid for all values of $\lambda$ except the zeros of $D_n(\lambda^n)$. It is evident that $K(x, y ; \lambda)$ is again the quotient of two entire functions, of which the denominator is $D_n(\lambda^n)$:

(44**) $K(x, y; \lambda) =$

$$\frac{D_n(\lambda^n)S(x,y;\lambda)+\lambda^{n-1}D_n\binom{x}{y};\lambda^n\Big)+\lambda^n\int_a^b S(x,\xi;\lambda)D_n\Big(\binom{\xi}{y};\lambda^n\Big)d\xi}{D_n(\lambda^n)}.$$

Now that this expression for the resolvent has been found, the Fredholm theorems may be applied to our more general case. Hence for example, when $\lambda$ is not a zero of $D_n(\lambda^n)$, the non-homogeneous integral equation has a unique solution which is still given by (II). It is merely necessary to substitute (44) for $K(x, y ; \lambda)$. Also the other theorems of Fredholm remain valid. We will not consider them in detail.

*Remark*: The Fredholm theorems concerning integral equations of the second kind remain valid when $x$ and $y$ are interpreted as points in a finite region of more than one dimension. The integrals become double, triple, or multiple.

# CHAPTER XI

## GENERAL SOLUTION OF THE BOUNDARY VALUE PROBLEMS

### Art. 1. Reduction to Integral Equations

In this chapter we will give the general solution of the boundary value problems formulated in Chapter VII. We will transform these problems to integral equations, and discuss the resulting equations.

First consider the Dirichlet problem for the interior region $V$ bounded by the closed surface $S$, which we will assume to have a continuous curvature. In the exterior problem, $S$ was supposed to consist of several parts, but for the interior problem the bounding surface $S$ is assumed to be a single surface, as this is no restriction on generality.[1] Now a harmonic function $u$ is sought, regular in $V$ and continuous in $V + S$ and satisfying the boundary condition

$$(1) \qquad u_- = f \text{ on } S,$$

where $f$ is a continuous function on $S$. (The symbols $u_+$, $u_-$, $\dfrac{\partial u_+}{\partial n}$, $\dfrac{\partial u_-}{\partial n}$ have the same meaning as in Chapter V, Art. 5 (after equation (16)) and Art. 6.)

Let

$$(2) \qquad u(P) = \iint_S \mu(Q) \frac{\partial\left(\frac{1}{r}\right)}{\partial n_Q} dS_Q = \iint_S \mu(Q) \frac{\cos(r, n_Q)}{r_{QP}^2} dS,$$

which is equivalent to assuming that $u$ is the potential due to a double layer on $S$ of moment $\mu$. Let us seek to determine $\mu$ so that the condition (1) is fulfilled. If this determination is

[1]See Chapter VII, Art. 3.

possible, then it is evident that (2) is the solution of the problem. For, the function $u$ defined by (2) clearly is a regular harmonic function in $V$, and if we complete the definition of $u$ by ascribing it the value $f$ on $S$, then from (1) it is continuous in $V + S$. Now we have shown[2] that on approaching a point $s$ of $S$ from the interior, the function defined by (2) satisfies the condition $u_- = u(s) - 2\pi\mu(s)$. Hence the moment $\mu$ must satisfy the equation

$$(3) \qquad u_- = -2\pi\mu(s) + \iint_S \mu(Q) \frac{\cos(r_{Qs}, n_Q)}{r_{Qs}^2} dS = f(s).$$

Let

$$(4) \qquad K(s, Q) = \frac{1}{2\pi} \frac{\cos(r_{Qs}, n_Q)}{r_{Qs}^2};$$

then the desired function is the solution of the equation

$$(5) \qquad \mu(s) - \iint_S K(s, Q)\mu(Q)\, dS = -\frac{f(s)}{2\pi},$$

a non-homogeneous integral equation of the second kind.

In the exterior problem, the condition to be satisfied is

$$(6) \qquad u_+ = f.$$

Using again the assumption (2), with $P$ now understood to be an exterior point, we obtain

$$u_+ = u(s) + 2\pi\mu(s), \text{ or } \mu(s) + \frac{u(s)}{2\pi} = \frac{f}{2\pi},$$

so that we need to solve the integral equation

$$(7) \qquad \mu(s) + \iint_S K(s, Q)\mu(Q)dS = \frac{f(s)}{2\pi}.$$

In the interior Neumann problem, we must satisfy the boundary condition

[2]See Chapter V, Art. 5, Equation (17), the notation $A$ for the surface point is changed to $s$.

$$\frac{\partial u_-}{\partial n} = f, \tag{8}$$

where it is necessary that the function $f$ satisfy the condition

$$\iint_S f dS = 0. \tag{9}$$

Here we let

$$u(P) = \iint_S \frac{\sigma dS}{r}, \tag{10}$$

i.e., assume that $u$ can be represented as the potential of a surface distribution of mass of density $\sigma$. Now (see Chapter V, Art. 6, equation (21*))

$$\frac{\partial u_-}{\partial n_s} = 2\pi\sigma(s) + \iint_S \sigma(Q) \frac{\partial\left(\dfrac{1}{r_{Qs}}\right)}{\partial n_s} dS,$$

or

$$\sigma(s) + \iint_S K(Q, s)\sigma(Q) dS = \frac{f(s)}{2\pi}. \tag{11}$$

Evidently

$$K(Q, s) = \frac{1}{2\pi} \frac{\cos(r_{Qs}, n_s)}{r_{Qs}^2} \tag{4*}$$

is the associated or transposed kernel to $K(s, Q)$ of (4).

For the exterior Neumann problem, with the boundary condition

$$\frac{\partial u_+}{\partial n} = f, \tag{12}$$

we use again the assumption (10) and obtain the integral equation

$$\sigma(s) - \iint_S K(Q, s)\sigma(Q) dS = -\frac{f(s)}{2\pi}. \tag{13}$$

Evidently the integral equations (11) and (13) are the associated or transposed equations corresponding to (7) and (5) respectively.

We can combine the integral equations for the first boundary problem in the form

$$\text{(I)} \qquad \mu(s) - \lambda \iint_S K(s, Q)\mu(Q)dS = g(s).$$

Here $\lambda$ is 1 for the interior problem and $-1$ for the exterior problem, and $g(s)$ is $-f(s)/2\pi$ and $f(s)/2\pi$ respectively.

Similarly, the solution for the second boundary problem is found by solving the integral equation

$$\text{(II)} \qquad \sigma(s) - \lambda \iint_S K(Q, s)\sigma(Q)dS = g(s)$$

with $\lambda$ equal to $-1$ and 1 respectively for the inner and outer problems, and $g(s)$ equal to $f(s)/2\pi$ and $-f(s)/2\pi$ respectively. The normal is assumed to be the outward normal on the closed surface or surfaces $S$ in every case.

For the interior third boundary value problem, we have

$$\text{(14)} \qquad \frac{\partial u_-}{\partial n} + \frac{h}{k} u_- = f,$$

where we suppose that the function $k$ does not vanish on $S$. We make again the assumption (10) used in the Neumann problem, and find

$$\frac{\partial u_-}{\partial n} + \frac{h}{k} u_- = 2\pi\sigma(s) + \iint \sigma(Q) \left\{ \frac{\cos(r, n_s)}{r^2} + \frac{h}{k} \cdot \frac{1}{r} \right\} dS = f(s),$$

or using the kernel

$$\text{(15)} \qquad \frac{1}{2\pi} \left\{ \frac{\cos(r, n_s)}{r_{sQ}^2} + \frac{h(Q)}{k(Q)} \frac{1}{r_{sQ}} \right\} = H(s, Q),$$

the resulting integral equation is

$$\sigma(s) + \iint_S H(s, Q)\sigma(Q)dS = \frac{f(s)}{2\pi}. \tag{16}$$

For the exterior third boundary value problem, we have

$$\frac{\partial u_+}{\partial n} + \frac{h}{k} u_+ = f, \tag{17}$$

and the resulting integral equation

$$\sigma(s) - \iint_S H(s, Q)\sigma(Q)dS = -\frac{f(s)}{2\pi} \tag{18}$$

for the density $\sigma$ to be used in (10).

The kernel $K(s, Q)$ of (I) becomes infinite as $Q \to s$, and i fact becomes infinite like $1/r_{Qs}$, since $\dfrac{\cos(r, n_Q)}{r}$ remains bounded on account of the continuous curvature of $S$. Since $K$ becomes infinite like $1/r$, it may be shown that the third iterated kernel $K^3(s, Q)$ remains bounded for all points $s$ and $Q$ of $S$, even when they coincide (see Art. 4). Hence the Fredholm theorems can be applied to (I) and also to (II). The kernel $H(s, Q)$ likewise becomes infinite like $1/r$ when $Q \to s$, so that $H^3(s, Q)$ is also bounded.

## Art. 2. The Existence Theorems

We can now prove: *The Dirichlet problem for the interior has always a unique solution.* That not more than one solution exists is already known, from Chapter VII. Now we will show that a solution actually exists. The homogeneous equation corresponding to (5) is

$$\mu(s) - \iint_S \mu(Q)K(s, Q)dS = 0. \tag{19}$$

It is sufficient to show that this has only the trivial solution 0;

for then by the Fredholm theorems, (5) must have a solution, and only one. Now let $\bar{\mu}$ be a solution of (19). Form the potential function $\bar{u}$ due to the double layer $\bar{\mu}$,

$$\bar{u} = \iint_S \bar{\mu} \frac{\cos(r, n)}{r^2} \, dS ;$$

then $\bar{u}_- = 0$, because $\bar{\mu}$ satisfies (19), and consequently $\bar{u} = 0$ identically in $V$ on account of the uniqueness theorem for the Dirichlet problem. Hence $\frac{\partial \bar{u}_-}{\partial n}$ on $S$ vanishes, and therefore $\frac{\partial \bar{u}_+}{\partial n} = 0$ (Chapter V; Art. 7). But from this it follows that $\bar{u} \equiv 0$ in the space outside $S$, from the uniqueness theorem for the exterior Neumann problem.[3] Hence $\bar{u}_+ = 0$ for all points of $S$. Hence finally

$$\bar{\mu}(s) = \frac{1}{4\pi} (\bar{u}_+ - \bar{u}_-) \equiv 0,$$

which completes the proof.

Hence 1 is not an eigen-value of (I). Accordingly, it is also not an eigen-value of the transposed equation (II), so that we also see immediately:

*The Neumann problem for the exterior of S always has a unique solution.*

The exterior Dirichlet problem, which is of a more complicated nature, will be considered in the next article.

We now consider the interior Neumann problem. We will show that the condition (9), which we know to be necessary, is also sufficient for the existence of a solution. It has already been seen that the solution, if it exists, is unique except for an arbitrary added constant (Chapter VII). The homogeneous equation corresponding to (11) is

[3] At first it follows that $\bar{u}$ = constant, but since $\bar{u} = 0$ at infinity, we get in fact $\bar{u} \equiv 0$.

$$\bar{\sigma}(s) + \iint\limits_S K(Q, s)\bar{\sigma}(Q)dS = 0. \tag{20}$$

The transposed or associated equation, which is the homogeneous form of (7), is

$$\bar{\mu}(s) + \iint\limits_S K(s, Q)\bar{\mu}(Q)dS = 0. \tag{20*}$$

The equation (20*) certainly has a non-trivial solution,

$$\bar{\mu} \equiv 1, \tag{21}$$

since from Chapter V, Art. 5,

$$\iint\limits_S \frac{\cos(r_{Qs}, n_Q)}{r_{Qs}^2} dS = -2\pi$$

holds identically for all points $s$ of $S$. We now assert that (20*) has no other solution not a constant. Let $\bar{\mu}$ be an arbitrary solution of (20*), and form the corresponding potential $\bar{u}$,

$$\bar{u} = \iint\limits_S \bar{\mu} \frac{\cos(r, n_Q)}{r^2} dS,$$

then

$$\bar{u}_+ = 2\pi\bar{\mu}(s) + \iint \bar{\mu}(Q) \frac{\cos(r, n_Q)}{r^2} dS = 0.$$

The harmonic function $\bar{u}$, being the potential of a double distribution, has the mass 0. It is therefore the solution of the exterior Dirichlet problem for $S$ with the mass 0 and the value 0 on $S$, and hence $\bar{u} \equiv 0$ in the entire exterior space, from the uniqueness theorem of Chapter VII, Art. 1. Therefore $\dfrac{\partial \bar{u}_+}{\partial n} = 0$ over the entire surface $S$, and hence $\dfrac{\partial \bar{u}_-}{\partial n} \equiv 0$ over $S$ also. Since the solution of the Neumann problem for the interior is unique to within an additive constant, $\bar{u}$ must be a

constant inside $S$, so that $\bar{u}_-$ is a constant on approaching $S$ from the interior. Hence finally, $\bar{\mu}(s) = \frac{1}{4\pi}(\bar{u}_+ - \bar{u}_-) =$ constant, as we wished to prove.

From the Fredholm theorems, it follows that the non-homogeneous equation (11) has a solution $\sigma$, because $f$ and hence also $f/2\pi$ is orthogonal to the only solution (a constant) of the transposed or associated homogeneous equation (20*), this being the condition (9) imposed on $f$. Using this solution $\sigma$ in (10), we obtain the solution $u$ of the interior Neumann problem. (The solution $\sigma$ of (11) is, moreover, only determined to within an additive constant, because (20*) has a constant as a solution.) Hence we have the theorem: *The Neumann problem for the interior of S has a solution satisfying* $\frac{\partial u_-}{\partial n} = f$ *if and only if* $\iint_S f dS = 0$; *the solution is uniquely determined except for an additive constant.*

From the above, it is evident that $-1$ is an eigen-value of (I) and hence also of (II).

We have reduced the third boundary value problem to the integral equations (16) or (18). The Fredholm theorems therefore lead to the following results: *Either the boundary value problem with the boundary condition* $\frac{\partial u_-}{\partial n} + \frac{h}{k} u_- = f$ *has a unique solution, or the corresponding homogeneous problem with the boundary condition* $\frac{\partial u_-}{\partial n} + \frac{h}{k} u_- = 0$ *has a finite number of linearly independent solutions.* In the last case, the non-homogeneous problem has in general no solution; it has a solution if and only if $f$ satisfies certain orthogonality conditions, and then the solution is no longer unique. Entirely analogous results hold for the exterior problem.

In the case of the heat conduction problem, where $\frac{h}{k} > 0$ always, it has already been shown (Chapter VII, Arts. 2, 3) that the homogeneous problem has no (non-trivial) solution. Hence it follows that: *The heat conduction problem with Newton's law of cooling has always a unique solution.*

## Art. 3. The First Boundary Value Problem for the Exterior

We will assume at first that the surface $S$ is a single closed surface; later the general case will be considered. In the preceding article, it was shown that $\lambda = -1$ is an eigen-value of the kernel $K(s, Q)$, and that the homogeneous equation (20*) has only the solution $\bar{\mu} = 1$ (to within a multiplicative constant). Accordingly, by the Fredholm theorems, (20) has only one solution $\bar{\sigma}$ (to within a constant factor). Form the potential

$$U(P) = \iint_S \frac{\bar{\sigma}(Q)}{r_{QP}}\, dS, \tag{22}$$

due to the surface distribution of mass of density $\bar{\sigma}$. On approaching $S$ from the interior, the normal derivative takes on the limiting value

$$\frac{\partial U_-}{\partial n_s} = 2\pi\bar{\sigma}(s) + \iint_S \bar{\sigma}(Q)\, \frac{\partial\left(\frac{1}{r_{Qs}}\right)}{\partial n_s}\, dS = 0,$$

because $\bar{\sigma}$ is a solution of (20). Therefore $\frac{\partial U_-}{\partial n} = 0$ for all points of $S$, and since the solution for the Neumann problem for the interior is unique to within an additive constant, it follows that $U$ must be a constant in the closed region $V_i + S$, say

$$U = C \text{ on } S \text{ and in} \tag{23}$$

If $S$ is an insulated electrical conductor which is given an electrical charge, then after equilibrium is reached the electrical potential is constant on $S$ and in $V_i$. The equation (23) is therefore characteristic for the *potential of an electrostatic conductor*. Hence (22) can be considered to be the potential due to a charge on a conductor, with $\bar{\sigma}$ representing the density of electricity on $S$. This density, from the above discussion, is a solution of (20), and hence an eigen-function for the kernel $K(Q, s)$. Since the solution $\bar{\sigma}$ has a constant factor at our disposal, we have at hand immediately the answer to the following questions:

a) What is the electrical charge density which creates the potential 1 on the conductor $S$?

Since the above density $\bar{\sigma}$ creates the potential $C$, the desired density is $\dfrac{\bar{\sigma}}{C}\cdot$

b) The total charge $E$ is placed on the conductor $S$. How does the charge distribute itself, or what is the density function?

For any particular charge density $\bar{\sigma}$, the total charge is $E_1 = \iint\limits_S \bar{\sigma} dS$. Then $\dfrac{E}{E_1}\bar{\sigma}$ is the desired density, because

$$\iint\limits_S \frac{E}{E_1}\bar{\sigma} dS = E.$$

We now return to the discussion of the Dirichlet problem. Still using the assumed form of solution (2), we are led to the equation (7). This non-homogeneous integral equation has a solution only if $f$ is orthogonal to $\bar{\sigma}$, that is, if $\iint\limits_S f(s)\bar{\sigma}(s) dS = 0$, which is not in general the case with an arbitrary function $f$. In fact, the potential for the exterior region $V_e$ cannot in general be expressed as the potential of a double layer. We

will now see that it is the sum of the potential of a double layer, a potential due to an electrostatic conductor, and a constant. The equation

$$(7_1) \quad \mu(s) + \iint_S K(s, Q)\mu(Q)ds = \frac{f(s) - C}{2\pi} \quad (C = \text{constant})$$

has a solution if and only if

$$\iint_S (f(s) - C)\bar{\sigma}(s)dS = 0,$$

or

$$(24) \qquad C = \frac{\iint_S f\bar{\sigma}dS}{\iint_S \bar{\sigma}dS}.$$

After $C$ has been determined in this way, $(7_1)$ has a solution; let $\mu$ be a solution. Then

$$(25) \qquad W = \iint_S \mu \frac{\cos(r, n)}{r^2} dS$$

is a regular harmonic function, which satisfies the boundary condition $W_+ = f(s) - C$ on approaching $S$ from the exterior region $V_e$.

Let $U$ be the potential which corresponds to a *unit* electrical charge on a conducting surface $S$,

$$(26) \qquad U = \iint_S \frac{\bar{\sigma}}{r} dS, \text{ with} \iint \bar{\sigma}dS = 1.$$

$U$ is evidently uniquely determined by $S$. For this potential $U$, let

$$(27) \qquad U = b \ (= \text{constant}) \text{ on } S.$$

Then $MU$ has the mass $M$ and takes on $S$ the constant value $Mb$. The function

$$w = W + MU \tag{28}$$

is therefore harmonic in the exterior $V_e$, has the required properties of regularity and has the prescribed mass $M$. Moreover, on approaching $S$,

$$w_+ = W_+ + MU_+ = f(s) - C + Mb.$$

Hence

$$u = W + MU + C - Mb \tag{29}$$

is the solution of the problem.

The generalization to the case where $S$ is composed of several separated surfaces $S_1, S_2, \ldots, S_m$ is now not difficult. The homogeneous equation

$$\bar{\mu}(s) + \iint_S K(s, Q)\mu(Q)dS = 0 \tag{20*}$$

now has the solutions

$$\bar{\mu}_k(s) = \begin{cases} 1 \text{ on } S_k, \\ 0 \text{ on } S_1, \ldots, S_{k-1}, S_{k+1}, \ldots, S_m \end{cases} \tag{30}$$
$$(k = 1, 2, 3, \ldots, m).$$

These $m$ solutions are evidently linearly independent. We will show that there are not more than $m$ linearly independent solutions of (20*). Let $\bar{\mu}$ be an arbitrary solution. Form the corresponding potential of a double layer

$$\bar{u} = \iint_S \bar{\mu} \frac{\cos(r, n)}{r^2} dS;$$

then from (20*) the equation

$$\bar{u}_+ = 2\pi\bar{\mu}(s) + \iint_S \bar{\mu}(Q) \frac{\cos(r, n_Q)}{r^2} dS = 0$$

holds over the entire surface $S$. The harmonic function $\bar{u}$ is therefore a solution of the exterior Dirichlet problem with the mass 0 and the boundary value $\bar{u}_+ = 0$. From the uniqueness theorem, $\bar{u} \equiv 0$ in $V_e$. Hence $\frac{\partial \bar{u}_+}{\partial n} = 0$ everywhere on $S$, and hence $\frac{\partial \bar{u}_-}{\partial n} \equiv 0$ also. By the Neumann interior problem uniqueness theorem, $\bar{u}$ is constant in the interior of each of the surfaces $S_k$. Hence $\bar{u}_-$ is constant over each of the surfaces $S_k$. Hence, finally, it follows that $\bar{\mu}(s) = \frac{1}{4\pi}(\bar{u}_+ - \bar{u}_-)$ is constant on each of the surfaces $S_k$; if these constants are $c_1, c_2, \ldots, c_m$, then evidently $\bar{\mu}$ can be written in the form

$$\bar{\mu} = c_1\mu_1 + c_2\mu_2 + \ldots + c_m\mu_m,$$

or is linearly dependent on the functions $\mu_k$, as we wished to prove.

Now, by the Fredholm theorems, the homogeneous equation

$$\bar{\sigma}(s) + \iint K(Q, s)\bar{\sigma}(Q)dS = 0 \tag{20}$$

has likewise $m$ linearly independent solutions and no more. Let

$$\bar{\sigma}_1(s), \bar{\sigma}_2(s), \ldots, \bar{\sigma}_m(s) \tag{31}$$

be an arbitrary system of linearly independent solutions. Form the corresponding potentials

$$U_k(P) = \iint_S \frac{\bar{\sigma}_k(Q)}{r} dS, \qquad k = 1, 2, \ldots m; \tag{32}$$

then the normal derivatives satisfy the conditions $\left(\frac{\partial U_k}{\partial n}\right)_- = 0$, because the $\bar{\sigma}_k$ are solutions of (20); the $U_k$ are constant inside

$S$ and hence on $S$. The potentials $U_k$ are therefore constant inside and on each of the surfaces $S_k$, or are potentials due to static charges on conductors $S_k$. The $U_k$ are linearly independent. For, if a relation $b_1U_1+\ldots+b_mU_m\equiv 0$ existed between them, then from the relations $4\pi\bar{\sigma}_k=\left(\dfrac{\partial U_k}{\partial n}\right)_- - \left(\dfrac{\partial U_k}{\partial n}\right)_+$ it would follow that the same relation $b_1\bar{\sigma}_1+\ldots+b_m\bar{\sigma}_m\equiv 0$ exists between the $\bar{\sigma}_k$; but these functions are linearly independent.

Every combination

$$\bar{U} = a_1U_1+\ldots+a_mU_m \tag{33}$$

is likewise constant on each of the surfaces $S_k$, and is hence the same as an electrostatic potential due to charged conductors. Evidently

$$\bar{U} = \iint\limits_S \frac{\tau(Q)}{r}\, dS, \tag{33*}$$

where

$$\tau = a_1\bar{\sigma}_1+\ldots+a_m\bar{\sigma}_m; \tag{34}$$

and $\tau$ is also a solution of (20). The equation (33), with arbitrary constant coefficients, gives the most general potential due to charged conductors $S_k$.

We now consider the problem of finding that conductor potential $\bar{U}$ which takes on prescribed values $C_1, C_2, \ldots, C_m$ on the corresponding surfaces. Let

$$U_k = \text{const.} = c_{kl} \text{ in } V_l+S_l, \tag{35}$$

where $V_k$ is the interior of $S_k$; i.e.

$$J_k = \begin{cases} c_{k1} \text{ in } V_1+S_1, \\ c_{k2} \text{ in } V_2+S_2, \\ \ldots\ldots\ldots\ldots \\ c_{km} \text{ in } V_m+S_m. \end{cases} \quad (k = 1, 2, \ldots m)$$

Assuming $\bar{U}$ in the form (33), we find for the $a_k$ the $m$ linear non-homogeneous equations

$$(36)\qquad \begin{cases} a_1c_{11} + a_2c_{21} + \ldots + a_mc_{m1} = C_1, \\ a_1c_{12} + a_2c_{22} + \ldots + a_mc_{m2} = C_2, \\ \cdots\cdots\cdots\cdots\cdots \\ a_1c_{1m} + a_2c_{2m} + \ldots + a_mc_{mm} = C_m. \end{cases}$$

The determinant $[c_{rs}]$ of the coefficients is not zero, since the potentials $U_1, U_2, \ldots, U_m$ are linearly independent. Hence the constants $a_1, a_2, \ldots, a_m$ are uniquely determined, and hence $\bar{U}$ is likewise uniquely determined. Naturally, the electrical density $\tau$ of $\bar{U}$ is also uniquely determined by (34). The problem is thus solved.

We now solve the following problem: Charges $E_1, E_2, \ldots, E_m$ are placed on the conducting surfaces $S_1, S_2, \ldots, S_m$. What is the potential $\bar{U}$ produced by them, and what is the density $\tau$?

The desired density $\tau$ must satisfy the equations

$$(37)\qquad \iint_{S_r} \tau dS = E_r \quad (r = 1, 2, \ldots, m).$$

By using (34), we again obtain $m$ linearly independent non-homogeneous equations for the constants $a_1, a_2, \ldots, a_m$. Let

$$(38)\qquad \bar{\sigma}_k = \sigma_{kl} \text{ on } S_l,$$

that is,

$$\bar{\sigma}_k = \begin{cases} \sigma_{k1} \text{ on } S_1, \\ \cdots\cdots \\ \sigma_{km} \text{ on } S_m \end{cases} \qquad (k = 1, 2, \ldots, m),$$

(the functions $\sigma_{rs}$ are naturally not constants), then the equations (37) take the form

$$(37^*)\quad a_1 \iint \sigma_{1r} dS + a_2 \iint_{S_r} \sigma_{2r} dS + \ldots + a_m \iint_{S_r} \sigma_{mr} dS = E_r \qquad (r = 1, 2, \ldots, m).$$

The non-homogeneous equations (37) or (37*) have a unique solution for the constants $a_1, a_2, \ldots, a_m$, since the corresponding homogeneous equations have no (non-trivial) solution. For, if a solution of the homogeneous equations existed, this would lead to a potential $\bar{U}$, not identically zero, produced by zero charges $E_r = 0$ on all the conductors, which is evidently impossible. (That this is impossible follows in a purely mathematical way from the Green's formula (30) in Chapter III, Art. 6.) Since the $a_k$ are uniquely determined, it follows that $\tau$ and $\bar{U}$ are uniquely determined, and the problem is solved.

We now return to the solution of the Dirichlet problem. The equation (7) of Art. 1 has, for an arbitrary function $f$, no solution. On the other hand, the equation

$$(7_2)\quad \mu(s) + \iint\limits_S K(s, Q)\mu(Q)dS = \frac{f(s) - c_1\mu_1 - c_2\bar{\mu}_2 - \ldots - c_m\bar{\mu}_m}{2\pi}$$

has a solution, if the constants $c_1, c_2, \ldots, c_m$ are determined so that the right member is orthogonal to all the $m$ solutions $\bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_m$ of the homogeneous equation (20). We must therefore determine the $c_1, c_2, \ldots, c_m$ to satisfy the $m$ equations

$$\iint\limits_S [f - c_1\bar{\mu}_1 - \ldots - c_m\bar{\mu}_m]\, \bar{\sigma}_k\, dS = 0$$

or

$$(39)\quad c_1\iint\limits_S \bar{\mu}_1\bar{\sigma}_k dS + c_2\iint\limits_S \bar{\mu}_2\bar{\sigma}_k dS + \ldots + c_m\iint\limits_S \bar{\mu}_m\bar{\sigma}_k dS$$
$$= \iint\limits_S f\bar{\sigma}_k dS,\ (k = 1, 2, \ldots, m).$$

The determinant of the coefficients of (39) is the same as that of (37*), as may be seen by noting equations (30) and (38). This determinant therefore is not zero, and the $c_k$ are uniquely

determined. With a solution $\mu$ of $(7_2)$, form the potential of a double layer

$$W = \iint_S \mu \frac{\cos(r, n)}{r^2} dS. \tag{40}$$

It has the mass zero, and the boundary value

$$W_\perp = f - c_1 \bar{\mu}_1 - \cdots - c_m \bar{\mu}_m$$

because $\mu$ satisfies the equation $(7_2)$.

We have already solved the problem of finding the conductor potential which takes on the prescribed values $C_1, C_2, \ldots, C_m$ on the corresponding surfaces. We now designate by $\bar{U}_k$ $(k = 1, 2, \ldots, m)$ that uniquely determined potential which takes on the value 1 on $S_k$ and vanishes on all the other surfaces. By (30), this is equivalent with the condition $(\bar{U}_k)_+ = \bar{\mu}_k$. Let $M_k$ designate the mass of the potential $\bar{U}_k$. Then we put the desired $u$ of the Dirichlet problem in the form

$$u = W + p_1\bar{U}_1 + p_2\bar{U}_2 + \ldots + p_m\bar{U}_m + c, \tag{41}$$

where the $p_1, p_2, \ldots, p_m$ and $c$ are constants. They must be determined so that $u_+ = f$ and the mass of $u$ is equal to the prescribed $M$. Hence

$$u_+ = f - c_1\mu_1 - \ldots - c_m\bar{\mu}_m + p_1\bar{\mu}_1 + \ldots + p_m\bar{\mu}_m + c = f \tag{42}$$

and

$$p_1M_1 + p_2M_2 + \ldots + p_mM_m = M. \tag{43}$$

From (42) it follows that everywhere on $S$,

$$(p_1 - c_1)\bar{\mu}_1 + \ldots + (p_m - c_m)\bar{\mu}_m + c = 0,$$

or by considering (30),

$$p_k - c_k + c = 0, \quad (k = 1, 2, \ldots, m)$$

or

$$p_1 = c_1 - c,\ p_2 = c_2 - c, \ldots, p_m = c_m - c. \tag{44}$$

Hence

$$(c_1 - c) M_1 + (c_2 - c) M_2 + \ldots + (c_m - c) M_m = M,$$

so that

$$c = \frac{c_1 M_1 + c_2 M_2 + \ldots + c_m M_m - M}{M_1 + M_2 + \ldots + M_m}. \tag{45}$$

Since the $c_1, c_2, \ldots, c_m$ were already fixed by (39), it is evident that $c$ is determined by (45) and then the $p_k$ by (44). Our problem is therefore solved. The solution is again the sum of the potential of a double distribution, a potential due to charges on insulated conducting surfaces and a constant.

If the Dirichlet problem for the exterior is put in the second simpler form (Chapter VII, Art. 3), the solution is of the form

$$u = W + c_1 \bar{U}_1 + \ldots + c_m \bar{U}_m. \tag{46}$$

Since evidently $u$ fulfils the requirements of regularity and continuity, and

$$u_+ = f - c_1 \bar{\mu}_1 - \ldots - c_m \bar{\mu}_m + c_1 \bar{\mu}_1 + \ldots + c_m \bar{\mu}_m = f,$$

*the solution is again uniquely determined.* For, if $u_2$ be a second solution it must have the form $u_2 = W_2 + U_2$, where $W_2$ is the potential of a double layer and $U_2$ is the potential of a simple distribution over $S$. Let $U_2$ have the mass $N_2$. Now $c_1 \bar{U}_1 + \ldots + c_m \bar{U}_m$ has the mass $c_1 M_1 + c_2 M_2 + \ldots + c_m M_m = N$.

Then
$$W_2 + U_2 - \frac{N_2}{N} (c_1 \bar{U}_1 + \ldots + c_m \bar{U}_m)$$

has the mass zero and takes on the boundary value

$$f - \frac{N_2}{N} (c_1 \bar{\mu}_1 + \ldots + c_m \bar{\mu}_m).$$

However, from the uniqueness theorem of Chapter VII, Art. 3, there is only one function of the form $w$ + const., where $w$ is

harmonic and regular in $V_e$, which has the mass zero and the above boundary value. This is

$$W + c_1\bar{\mu}_1 + \ldots + c_m\bar{\mu}_m - \frac{N_2}{N}(c_1\bar{\mu}_1 + \ldots + c_m\bar{\mu}_m).$$

Accordingly

$$W_2 + U_2 - \frac{N_2}{N}(c_1\bar{U}_1 + \ldots + c_m\bar{U}_m)$$
$$\equiv W + \left(1 - \frac{N_2}{N}\right)(c_1\bar{\mu}_1 + \ldots + c_m\bar{\mu}_m).$$

From this it follows that

$$1 - \frac{N_2}{N} = 0, \qquad \text{or } N_2 = N,$$

and that

$$W_2 \equiv W, \qquad \text{and } U_2 - \frac{N_2}{N}(c_1\bar{U}_1 + \ldots + c_m\bar{U}_m) \equiv 0$$

or

$$U_2 = c_1\bar{U}_1 + \ldots. + c_m\bar{U}_m,$$

which we wished to prove.

The solution given in Chapter IX, Art. 1, for the first boundary value problem for the exterior of the sphere, by means of the Poisson integral, is evidently the sum of the potential of a double layer and of the potential due to a charged conductor, and from the above results this solution is unique.

## Art. 4. Boundedness of the Third Iterated Kernel

To complete our proofs, we still have to demonstrate the assertion made above that the third iterated kernel $K^3(s, Q)$ is bounded.

For this purpose, we will first prove an auxiliary theorem. Let $T$ be a bounded *plane* region, $s$ and $w$ points of $T$, and let $g(s, w)$ be a function such that for all points $s$, $w$ in $T$,

$$g(s, w) = \frac{G(s, w)}{r_{sw}^{\alpha}},$$

where $r_{sw}$ is the distance from $s$ to $w$, and $0 < \alpha < 2$, and where $G(s, w)$ is bounded for all points $s$, $w$, or $|G(s, w)| < M$. Likewise, let

$$h(s, w) = \frac{H(s, w)}{r_{sw}^{\beta}}$$

where $0 < \beta < 2$, $|H(s, w)| < M$.

Let

$$f(s,w) = \iint\limits_{T} g(s, w_1)\, h(w_1, w)\, dS_{w_1} = \iint\limits_{T} \frac{G(s, w_1)H(w_1, w)}{r_{sw_1}^{\alpha}\; r_{w_1 w}^{\beta}}\, dS_{w_1};$$

we will investigate the behaviour of $f(s, w)$ when $w \to s$.

It is known that the integral

$$\iint\limits_{T} \frac{dxdy}{r^{\alpha}},$$

where $r$ is the distance of the point $(x, y)$ from the origin, has a definite value for $\alpha < 2$ even when the origin lies in $T$. This is proved easily by introducing polar coordinates, so that

$$\iint\limits_{T} \frac{dxdy}{r^{\alpha}} = \iint \frac{drd\phi}{r^{\alpha-1}},$$

and the exponent $\alpha - 1 < 1$.

From this it follows that $f(s, w)$ is bounded as long as $w$ remains at a finite distance from $s$.

When $w$ approaches indefinitely near to $s$, the integral $f(s, w)$ still remains bounded if $\alpha + \beta - 2 < 0$, because the integral

$$\iint\limits_{T} \frac{G(s, w_1)H(w_1, w)}{r_{sw_1}^{\alpha+\beta}}\, dS_{w_1}$$

still has a value.

However, it is different when $\alpha+\beta-2 \geq 0$. We claim that in the case $\alpha+\beta-2>0$, the function

$$r_{sw}^{\alpha+\beta-2} f(s, w)=F(s, w)$$

remains bounded, and therefore that $f(s, w)$ may become infinite of the order $\alpha+\beta-2$ with respect to $1/r_{sw}$, and that for $\alpha+\beta-2=0$, $f(s, w)$ may become infinite like $\log(1/r)$.

To prove this, consider $s$ as fixed and as the centre of a circle of radius $2r_{sw}$; let $T_1$ be the interior of this circle, and $T_2$ be the remainder of the region $T$. Then

$$f(s, w)=\iint_T \cdots=\iint_{T_1} \cdots+\iint_{T_2} \cdots$$

As long as the integration point $w_1$ lies in $T_2$, the quotient $\frac{r_{w w_1}}{r_{s w_1}}$ lies between two positive bounds, $0<a<\frac{r_{w w_1}}{r_{s w_1}}<b$, so that one can replace $r_{s w_1}$ by $r_{w w_1}$, or vice versa, in any inequality. Hence

$$\left|\iint_{T_2} \frac{G(s, w_1) H(w_1, w)}{r_{s w_1}^{\alpha} r_{w w_1}^{\beta}} d S\right|<\frac{M^2}{a^\beta} \iint_{T_2} \frac{d S}{r_{s w_1}^{\alpha+\beta}} .$$

Introducing polar coordinates with $s$ as pole, then for $\alpha+\beta-2>0$

$$\iint_{T_2} \frac{d S}{r_{s w_1}^{\alpha+\beta}}<2 \pi \int_{2 r_{s w}}^R \frac{d r}{r^{\alpha+\beta-1}}=\left[\frac{-2 \pi}{(\alpha+\beta-2) r^{\alpha+\beta-2}}\right]_{r=2 r_{s w}}^{r=R}$$

where $R$ is large enough so that $T_2$ lies in the ring $2 r_{s w} \leq r \leq R$. Hence

$$\iint g(s, w_1) h(w_1, w)\, d S=O\left(\frac{1}{r_{s w}^{\alpha+\beta-2}}\right) .$$

For $\alpha+\beta-2=0$, since $\int \frac{d r}{r}=\log r$, it is evident that the

integral $\iint_{T_2} g(s, w_1)h(w_1, w)dS_{w_1}$ may become infinite like $\log(1/r_{sw})$.

For the other integral,

$$\left|\iint_{T_1} g(s, w_1)h(w_1, w)\, dS_{w_1}\right| < M^2 \iint_{T_1} \frac{dS}{r_{sw_1}^{\alpha}\, r_{w_1w}^{\beta}}\,.$$

We introduce a similarity transformation which enlarges $T_1$ to the unit circle. If for brevity $2r_{sw} = k$, then denoting the points in the new coordinate system by bars, we have

$$r_{sw_1} = kr_{\overline{sw_1}}\,,\quad r_{w_1w} = kr_{\overline{w_1w}}\,,\quad dS_{w_1} = k^2 d\bar{S}_{\overline{w_1}}$$

and

$$\iint_{T_1} \frac{dS_{w_1}}{r_{sw_1}^{\alpha}\, r_{w_1w}^{\beta}} = \frac{1}{k^{\alpha+\beta-2}} \iint_{\binom{\text{unit}}{\text{circle}}} \frac{d\bar{S}}{r_{\overline{sw_1}}^{\alpha}\, r_{\overline{w_1w}}^{\beta}}\,.$$

The integral on the right has a value independent of the position of the points $s$ and $w$, and consequently remains fixed when $w \to s$. Hence for $\alpha + \beta - 2 > 0$

$$\iint_{T_1} g(s, w_1)h(w_1, w)\, dS_{w_1} = O\left(\frac{1}{k^{\alpha+\beta-2}}\right) = O\left(\frac{1}{r_{sw}^{\alpha+\beta-2}}\right),$$

and for $\alpha + \beta - 2 = 0$, the integral $\iint_{T_1} \ldots$ remains bounded.

This completes the proof.

The theorem just proved remains valid if the plane region $T$ is replaced by a curved surface $S$, which has continuous curvature, as for example the surface $S$ used in the boundary value problems. For we may think of the tangent plane being drawn at the point $s$ of $S$, and separate $S$ into a small region $S'$ about this point with $S''$ being the remainder of the surface; then project $S'$ on this tangent plane to form

the region $T$ there. There is then a one-to-one correspondence between points of $S'$ and of $T$. The ratio of the distance $r_{sw}$ of a point $w$ of $S'$ from $s$, to the corresponding distance between the corresponding points in the tangent plane, is bounded away from zero (on account of the continuous curvature of the surface $S$), and this is also true for corresponding elements of integration. Hence we have first

$\int_T \ldots = O\left(\frac{1}{r_{sw}^{\alpha+\beta-2}}\right)$; and from this that $\iint_{S'} \ldots = O\left(\frac{1}{r_{sw}^{\alpha+\beta-2}}\right)$;

while the remaining part of the integral $\iint_{S''} \ldots\ldots$ remains bounded, so that finally

$$\iint_S \ldots . = O\left(\frac{1}{r_{sw}^{\alpha+\beta-2}}\right)$$

when $\alpha + \beta - 2 > 0$. In the case $\alpha + \beta - 2 = 0$, the integrals $\iint_T \ldots$ and finally $\iint_S \ldots .$ may become logarithmically infinite as $w \to s$.

Having completed this auxiliary theorem; we will have no difficulty in proving that $K^3(s, Q)$ is bounded.

First consider

$$K^2(s, Q) = \iint_S K(s, w)K(w, Q)\, dS_w.$$

Since we can set

$$K(s, Q) = \frac{G(s, Q)}{r_{sQ}},$$

where $G(s, Q)$ is a bounded function, we have

$$\alpha = 1, \beta = 1 \;;\; \alpha + \beta - 2 = 0.$$

Hence $K^2(s, Q)$ becomes at most logarithmically infinite as $Q \to s$. Then, since

$$K^3(s, Q) = \iint_S K(s, w)K^2(w, Q)\, dS_w,$$

we have in this case that $\alpha = 1$ while $\beta$ can be an arbitrarily small positive constant. Hence, since $\alpha + \beta - 2 < 0$, the kernel $K^3(s, Q)$ remains bounded for all points $s$ and $Q$.

*Exercises*:

Formulate the boundary value problems for logarithmic potential.

Prove the uniqueness theorems.

Reduce the problems to integral equations.

Prove the existence theorems.

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